Math 3204: Calculus IV[1] Mikhail Lavrov

Lecture 11: Identifying conservative vector fields

September 17, 2024 *Kennesaw State University*

1 Conservative vector fields in $\mathbb{R}^2$

The story so far: we know that if a vector field $\mathbf{F}$ is the gradient $\boldsymbol{\nabla} f$ of some scalar function f, then $\mathbf{F}$ is conservative, which means that line integrals of $\mathbf{F}$ are path independent, and we can even compute them just by evaluating f at the endpoints.

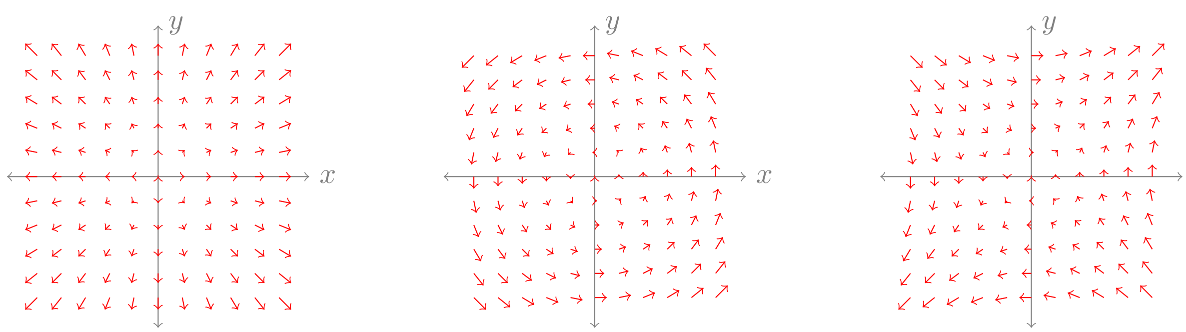
There's just a few problems:

- In order to be able to make use of this in a non-contrived problem, we would have to be able to tell when $\mathbf{F}$ is a gradient field.
- Moreover, given a gradient field $\mathbf{F}$, we would need a way to find a function f such that $\mathbf{F} = \boldsymbol{\nabla} f$. (Such a function is called a **potential function** for $\mathbf{F}$.)

We can already use some techniques from the previous lecture to answer these problems, but they are rather indirect; we can do things in a more straightforward way.

1.1 Some special cases

Let's go back to the three simple vector fields we looked at a few lectures ago (Figure 1.)



(a) $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$ (b) $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}$ (c) $\mathbf{F} = y\,\mathbf{i} + x\,\mathbf{j}$

Figure 1: Three simple vector fields in $\mathbb{R}^2$

Of these, the vector field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$ (Figure 1a) is the easiest one to deal with. As single-variable functions, the antiderivative of x is $\frac{x^2}{2}$ and the antiderivative of y is $\frac{y^2}{2}$. So if $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$, then $\frac{\partial f}{\partial x} = x$ and $\frac{\partial f}{\partial y} = y$, so $\boldsymbol{\nabla} f = x\,\mathbf{i} + y\,\mathbf{j} = \mathbf{F}$.

[1]This document comes from an archive of the Math 3204 course webpage: http://misha.fish/archive/3204-fall-2024

In fact, more is true. Every vector field $\mathbf{F}$ of the form $g(x)\,\mathbf{i} + h(y)\,\mathbf{j}$ is a gradient field (and a similar fact holds in $\mathbb{R}^n$). We just need to find antiderivatives G and H such that $G'(x) = g(x)$ and $H'(y) = h(y)$; then $f(x, y) = G(x) + H(y)$ is a potential function for $\mathbf{F}$.

It might not be as obvious that $y\,\mathbf{i} + x\,\mathbf{j}$ is the gradient of xy, while $-y\,\mathbf{i} + x\,\mathbf{j}$ is not the gradient of any scalar function.

1.2 The component test

We will need a result called Clairaut's theorem, which I will state here in full generality.

Theorem 1.1 (Clairaut's theorem)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function of n variables $x_1, x_2, \ldots, x_n$, which has continuous second partial derivatives. Then for every pair of variables x_i, x_j, and at every point $\mathbf{p}$, we have*

$$\frac{\partial^2 f}{\partial x_i\,\partial x_j}(\mathbf{p}) = \frac{\partial^2 f}{\partial x_j\,\partial x_i}(\mathbf{p})$$

Moreover, it's enough for f to be defined, and for the hypotheses to hold, within a small neighborhood of $\mathbf{p}$ (within a ball around $\mathbf{p}$ of arbitrarily small radius).

We will not prove Theorem 1.1, though it can be proven using some tools we'll develop later this semester.

In $\mathbb{R}^2$, the theorem says that for any function f of x and y, if we take the partial derivative with respect to both x *and* y, then we get the same result no matter which order we do it in. For example, suppose $f(x, y) = x^3y + 2y^3$. Then

- The partial derivative $\frac{\partial f}{\partial x}$ is $3x^2y$, and the partial derivative of that with respect to y is $\frac{\partial^2 f}{\partial y\,\partial x} = 3x^2$.
- The partial derivative $\frac{\partial f}{\partial y}$ is $x^3 + 6y^2$, and the partial derivative of that with respect to x is $\frac{\partial^2 f}{\partial x\,\partial y} = 3x^2$.

This result does not *always* hold, but it holds provided that f is sufficiently nice, and Theorem 1.1 is one way to specify what "sufficiently nice" means. Even if f is not nice everywhere (or even defined everywhere), we can still apply the theorem a little bit away from the points where f misbehaves.

How are the second partial derivatives relevant to us?

Well, suppose we have $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}$, and we're wondering whether some function $f(x, y)$ exists such that $\frac{\partial f}{\partial x} = -y$ and $\frac{\partial f}{\partial y} = x$. Well, if it did, then we'd be able to compute:

$$\begin{aligned}
\frac{\partial^2 f}{\partial y\,\partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(-y) = -1, \\
\frac{\partial^2 f}{\partial x\,\partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x) = 1.
\end{aligned}$$

These are not the same! But Theorem 1.1 says they have to be: and the only hypothesis of the theorem that could possibly be violated here is the existence of f, since we can tell that -1 and 1 are continuous functions defined everywhere.

In full generality, we have a fact called the **component test**:

Theorem 1.2. *If $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ is a gradient vector field, then* $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

So if we take these partial derivatives, and we don't get matching results, we know that $\mathbf{F}$ can't possibly be a gradient vector field?

What if we do get matching results? Well, the theorem says "if", not "if and only if". Drawing the reverse conclusion is slightly more complicated; we can only do it sometimes, and it will take a bit before we can explain why.

Still, if we can *find* a potential function f such that $\mathbf{F} = \boldsymbol{\nabla} f$, then that's definitive. Howe can we do that?

1.3 Finding a potential function

Finding a potential function is a lot like finding an antiderivative, but it is more complicated when we have multiple variables.

For single-variable functions, antiderivatives are indefinite integrals; you may remember that in single-variable calculus, we make a big deal out of including a $+C$ term. For example,

$$\int \sin 2x \,\mathrm{d}x = -\frac{1}{2}\cos 2x + C.$$

Sometimes this $+C$ term is actually relevant. For example, if you find the integral above by writing $\sin 2x$ as $2\sin x\cos x$, then doing a u-substitution with $u = \sin x$ and $\mathrm{d}u = \cos x\,\mathrm{d}x$, we will get

$$\int 2\sin\cos\,\mathrm{d}x = \int 2u\,\mathrm{d}u = u^2 + C = \sin^2 x + C.$$

What happened. Did we break calculus? No, it's just that $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$. So we got two antiderivatives that are off by a constant; that's *why* the $+C$ term is necessary.

When finding potential functions, the $+C$ term becomes even more important. That's because a partial derivative $\frac{\partial f}{\partial x}$ treats all variables other than x as constants—therefore, when we find the antiderivative, the $+C$ term (an arbitrary "constant" term) can include any variables other than x in it.

Here's an example. Suppose $\mathbf{F} = 2xy^3\,\mathbf{i} + 3(x^2-1)y^2\,\mathbf{j}$. We want to know if there is a function $f(x, y)$ such that $\mathbf{F} = \boldsymbol{\nabla} f$.

In particular, this means that $\frac{\partial f}{\partial x} = 2xy^3$. We can try to recover f by taking the antiderivative with respect to x:

$$\int 2xy^3\,\mathrm{d}x = x^2y^3 + C.$$

Here, the $+C$ term can hide any function of y. So all we know is: if $\frac{\partial f}{\partial x} = 2xy^3$, then $f(x, y) = x^2y^3 + g(y)$ for some single-variable function g.

To figure out what g is, let's use our partial information to find $\frac{\partial f}{\partial y}$: we have

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2y^3 + g(y)) = 3x^2y^2 + \frac{\mathrm{d}g}{\mathrm{d}y}.$$

But we are hoping to get $\frac{\partial f}{\partial y} = 3(x^2-1)y^2$. Setting these equal, we get

$$3x^2y^2 + \frac{\mathrm{d}g}{\mathrm{d}y} = 3(x^2-1)y^2 \implies \frac{\mathrm{d}g}{\mathrm{d}y} = -3y^2.$$

At this point, we are pleased to see that the expression we got for $\frac{\mathrm{d}g}{\mathrm{d}y}$ only depends on y. If the right-hand side still had x in it, we would conclude that there's no potential function; this is equivalent to the component test.

As it is, taking another antiderivative, we conclude that $g(y) = -y^3 + C$. We can put everything together and get $f(x,y) = x^2y^3 - y^3 + C$. As C varies, this gives us all possible potential functions for $\mathbf{F} = 2xy^3\,\mathbf{i} + 3(x^2-1)y^2\,\mathbf{j}$.

(By the way, if this process for finding the potential function *also* tells us if $\mathbf{F}$ is a gradient field, why do we need the component test? Well, in simple examples, we don't. In general, however, finding antiderivatives is much harder than taking derivatives—so the component test can be used to tell us when we don't need to bother.)

2 Conservative fields in $\mathbb{R}^3$

What we've done in two dimensions continues to work in three dimensions, but the calculations become more complicated.

2.1 The component test

If we're trying to test the hypothesis that $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is equal to $\boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}$ for some f, then there are three partial derivatives we need to check.

Theorem 2.1. *If $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is a gradient field, then*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}.$$

Proof. If $\mathbf{F} = \boldsymbol{\nabla} f$ for some scalar function $f\colon \mathbb{R}^3 \to \mathbb{R}$, then $\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}$ and $\frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}$, so they should be equal. (This is the same check as we did in two dimensions.

Similarly, we should have

$$\frac{\partial M}{\partial z} = \frac{\partial^2 f}{\partial z\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial z} = \frac{\partial P}{\partial x}$$

and

$$\frac{\partial N}{\partial z} = \frac{\partial^2 f}{\partial z\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial z} = \frac{\partial P}{\partial y}$$

which gives us the other two conditions we need to check. □

Let's try it on an example: $\mathbf{F} = y^2\,\mathbf{i} + 2xy\,\mathbf{j} + y^3\,\mathbf{k}$:

- We check $\frac{\partial}{\partial y}(y^2) = 2y$ and $\frac{\partial}{\partial x}(2xy) = 2y$, so the first condition holds.
- We check $\frac{\partial}{\partial z}(y^2) = \frac{\partial}{\partial x}(y^3) = 0$, so the second condition holds.

- However, $\frac{\partial}{\partial z}(2xy) = 0$ while $\frac{\partial}{\partial y}(y^3) = 3y^2$, so the third condition does not hold. $\mathbf{F}$ is not conservative: it is not a gradient field.

2.2 Finding a potential function

Let's see how things go if we want to find a potential function for the three-dimensional vector field

$$\mathbf{F} = (2xy + 3z^2)\,\mathbf{i} + (x^2 + 4z^2)\,\mathbf{j} + (6xz + 8yz)\,\mathbf{k}.$$

This is similar to our two-dimensional example, but with an extra layer of complexity.

1. Suppose that there is a function $f\colon \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F} = \boldsymbol{\nabla} f$. Then, in particular, $\frac{\partial f}{\partial x} = 2xy + 3z^2$. Taking the antiderivative gives us $x^2y + 3xz^2$, up to a constant; therefore we must have $f(x, y, z) = x^2y + 3xz^2 + g(y, z)$ for some function g.

2. Using this assumption, we compute $\frac{\partial f}{\partial y}$ and set it equal to $x^2 + 4z^2$. We get

$$x^2 + \frac{\partial g}{\partial y} = x^2 + 4z^2 \implies \frac{\partial g}{\partial y} = 4z^2$$

   which implies that $g(y, z) = 4yz^2 + h(z)$ for some function h, and $f(x, y, z) = x^2y + 3xz^2 + 4yz^2 + h(z)$.

3. Using this assumption, we compute $\frac{\partial f}{\partial z}$ and set it equal to $6xz + 8yz$. We get

$$0 + 6xz + 8yz + \frac{\mathrm{d}h}{\mathrm{d}z} = 6xz + 8yz \implies \frac{\mathrm{d}h}{\mathrm{d}z} = 0,$$

   so $h(z)$ must actually be a constant.

Our final answer is $f(x, y, z) = x^2y + 3xz^2 + 4yz^2 + C$, where C can be any constant.