Math 3204: Calculus IV[1]	Mikhail Lavrov

Lecture 14: Green's theorem

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1 Green's theorem for circulation

1.1 Integrating circulation density

In the previous section, we defined the curl, or circulation density, of a vector field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ to be $\operatorname{curl}\mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$. This was motivated by our calculations that showed that the counterclockwise circulation along a tiny loop around (x, y) is approximately equal to the circulation density at (x, y), multiplied by the area inside the loop.

Of the two names we have for this quantity, "curl" is definitely the more standard, but "circulation density" is the one that's more suggestive of what we're about to do with it:

Theorem 1.1 (Green's theorem for circulation)**.** *Let R be a region in $\mathbb{R}^2$, and let C be the boundary of R, oriented counterclockwise. Let $\mathbf{F}$ be a vector field.*

Suppose that all three of R, C, and $\mathbf{F}$ are well-behaved in their own way: R is bounded and simply connected (no holes), C is a smooth or at least piecewise smooth curve, and $\mathbf{F}$ is defined and has continuous partial derivatives on an open set containing R.

Then the circulation of $\mathbf{F}$ along C is equal to the integral of the circulation density of $\mathbf{F}$ over R:

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_R \operatorname{curl}\mathbf{F}\,\mathrm{d}x\,\mathrm{d}y.$$

Just like we find the mass of a region by integrating density over that region, we find the circulation around R by integrating the circulation density over R.

This is an incredibly valuable theorem because it lets us turn a line integral into a completely different kind of integral! Sometimes the line integral will be easier to do, and sometimes the double integral will be easier to do. Either way, having the choice between the two options is very useful.

1.2 The proof of Green's theorem

A simple but important fact about double integrals $\iint_R$ is that when R_1 and R_2 are disjoint regions,

$$\iint_{R_1 \cup R_2} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{R_1} f(x, y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{R_2} f(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

That is, if we combine two non-overlapping regions into one, the integrals add.

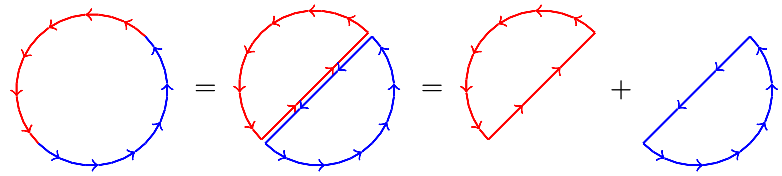
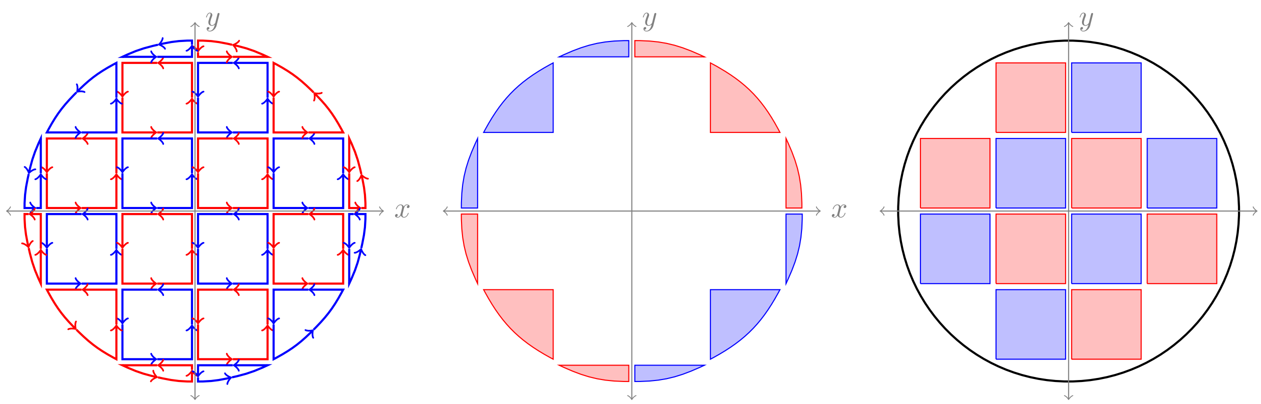


Figure 1: Adding circulation integrals

In fact, the same thing is true for circulation integrals around the boundaries of those regions! This is convincing evidence that Green's theorem *could* be true, and will be important in its proof.

Figure 1 shows an example of this. Suppose we separate a circle into two half-circles, and take the counterclockwise circulation around each half-circle. Each half-circle has two boundaries: a curved arc, and a diameter. When we add the circulation, the curved arcs join together into the boundary of the full circle. The diameters, in the meantime, are equal but have opposite orientation, so the circulation along the diameters cancels. We are left only with the circulation around the full circle.

For the proof of Green's theorem, we look at a different decomposition of our region.



(a) A subdivision of the region into small cells

(b) The fragmented cells have small area and can be ignored

(c) Circulation around each square cell contributes curl

Figure 2: An illustration of the proof of Green's theorem

Taking many horizontal and vertical slices, we divide the region R into many small cells, as shown in Figure 2a using a circle as an example. The circulation around R is (exactly!) equal to the sum of the circulations around every small cell.

There are two types of small cells. Let's begin by looking at the cells we'll ultimately want to discard: the fragmented cells around the boundary of R, shown in Figure 2b.

We haven't proven almost anything about how circulation around a non-square region behaves; that's what Green's theorem is for. However, we do have an argument that as a region shrinks, the circulation around its boundary goes to 0 proportionally to the *area* of that region. The total

[1]This document comes from an archive of the Math 3204 course webpage: http://misha.fish/archive/3204-fall-2024

area of the fragmanted cells will approach 0 as our division of R gets finer and finer; therefore the contribution to the circulation around R from the fragmented cells will also approach 0.

In other words, we have

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} \approx \sum_{i=1}^{N} \int_{\square_i} \mathbf{F} \cdot \mathrm{d}\mathbf{r}, \tag{1}$$

where C is the counterclockwise boundary of R, and $\square_1, \dots, \square_N$ are the counterclockwise boundaries of the N square cells—the cells shown in Figure 2c. We can make this approximation arbitrarily good by making the cells smaller.

This means it's time to look at what these square cells are doing!

From the previous lecture, we know that the circulation around a small square cell is approximately equal to the curl of $\mathbf{F}$ at the center, multiplied by its area. Let (x_i, y_i) be the center of the i^{th} square cell, and let ΔA be the area of each square cell. Then we have

$$\sum_{i=1}^{N} \int_{\square_i} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \sum_{i=1}^{N} \frac{\int_{\square_i} \mathbf{F} \cdot \mathrm{d}\mathbf{r}}{\Delta A} \cdot \Delta A \approx \sum_{i=1}^{N} \operatorname{curl} \mathbf{F}(x_i, y_i) \cdot \Delta A. \tag{2}$$

This is a Riemann sum for a double integral: we have

$$\sum_{i=1}^{N} \operatorname{curl} \mathbf{F}(x_i, y_i) \cdot \Delta A \approx \iint_R \operatorname{curl} \mathbf{F} \, \mathrm{d}x \, \mathrm{d}y. \tag{3}$$

Chaining together (1), (2), and (3), we learn that the two sides of Green's theorem are both at least *approximately* equal:

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} \approx \iint_R \operatorname{curl} \mathbf{F} \, \mathrm{d}x \, \mathrm{d}y$$

To show they are *exactly* equal, we take the limit as $\Delta A \to 0$ and $N \to \infty$, where all three approximations become arbitrarily precise.

We can now think about where the conditions on our problem come from: how could this proof go wrong?

- If R is not simply connected—it has holes—then its boundary might be more complicated than a single curve C.
- There exist regions R with boundaries so terrible that in fact a significant fraction of the area of R is arbitrarily close to a boundary. In that case, the fragmented cells in Figure 2b might not actually be negligible.

  To avoid this sort of behavior, we assume that C is smooth or at least piecewise smooth (though a weaker hypothesis would be enough).
- If $\mathbf{F}$ does not have continuous partial derivatives on every tiny cell, then the circulation around a cell might not be equal to a circulation density multiplied by an area. (Consider the vector field coming from the $\mathrm{d}\theta$ differential form: here, an arbitrarily tiny cell centered at $(0, 0)$ will have a circulation of 2π.)

1.3 An example

Let $\mathbf{F} = x\,\mathbf{i} + xy\,\mathbf{j}$. Let C be the curve made up of two segments:

$$\begin{aligned} \mathbf{r}_1(t) &= (t, -2) & t &\in [-2, 2] \\ \mathbf{r}_2(t) &= (-t, 2 - t^2) & t &\in [-2, 2] \end{aligned}$$

Suppose that we want to take the circulation integral of $\mathbf{F}$ around C.

To apply Green's theorem, we need to first realize that C is the counterclockwise boundary of the region

$$R = \{(x, y) \in \mathbb{R}^2 : -2 \le x \le 2 \text{ and } -2 \le y \le 2 - x^2\}.$$

The limits on y are the line $y = -2$ and $y = 2 - x^2$, which intersect at $(-2, 2)$ and $(2, 2)$, and these are also the bounds on x. The portion of C parameterized by $\mathbf{r}_1(t)$ is the straight line segment going from $(-2, 2)$ to $(2, 2)$ along the boundary $y = -2$. The portion of C parameterized by $\mathbf{r}_2(t)$ is the parabolic curve going back from $(2, 2)$ to $(-2, 2)$ along the boundary $y = 2 - x^2$.

Now we have a description of R we can use to integrate. We also need the circulation density of $\mathbf{F}$: it is $\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}x = y$. Now we can integrate:

$$\begin{aligned} \iint_R \operatorname{curl} \mathbf{F} \, \mathrm{d}x \, \mathrm{d}y &= \int_{x=-2}^{2} \int_{y=-2}^{2-x^2} y \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_{x=-2}^{2} \left. \frac{y^2}{2} \right|_{y=-2}^{2-x^2} \mathrm{d}x \\ &= \int_{x=-2}^{2} \left(\frac{(2 - x^2)^2}{2} - \frac{(-2)^2}{2} \right) \mathrm{d}x \\ &= \int_{x=-2}^{2} \left(\frac{1}{2}x^4 - 2x^2 \right) \mathrm{d}x \\ &= \left. \frac{x^5}{10} - \frac{2x^3}{3} \right|_{x=-2}^{2} \\ &= \frac{32}{10} - \frac{16}{3} - \left(-\frac{32}{10} + \frac{16}{3} \right) = -\frac{64}{15}. \end{aligned}$$

This should match the answer we get by computing the circulation along the two segments parameterized by $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$, then adding the results together.

2 Other versions of Green's theorem

2.1 Green's theorem for flux

There is also a version of Green's theorem for flux. The key quantity in this case is the flux density, or divergence, of $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$:

$$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

Theorem 2.1 (Green's theorem for flux)**.** *Let R be a region in $\mathbb{R}^2$, and let C be the boundary of R, oriented counterclockwise. Let $\mathbf{F}$ be a vector field.*

As before, suppose that all three of R, C, and $\mathbf{F}$ are well-behaved in their own way: R is bounded and simply connected (no holes), C is a smooth or at least piecewise smooth curve, and $\mathbf{F}$ is defined and has continuous partial derivatives on an open set containing R.

Then the outward flux of $\mathbf{F}$ across C is equal to the integral of the circulation density of $\mathbf{F}$ over R:

$$\int_C \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s = \iint_R \operatorname{div} \mathbf{F} \, \mathrm{d}x \, \mathrm{d}y.$$

The proof is essentially the same, because when we combine two non-overlapping regions, the outward flux across their boundaries *also* adds! When the two regions share a boundary, the "outward" direction for one of the regions will be equal to the "inward" direction for the other region.

Keeping the same $\mathbf{F}$ and C as in subsection 1.3, let's compute the outward flux of $\mathbf{F}$ across C.

We begin with computing the flux density: $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(xy) = 1 + x$. Now we can integrate:

$$\begin{aligned}
\iint_R \operatorname{div} \mathbf{F} \, \mathrm{d}x \, \mathrm{d}y &= \int_{x=-2}^{2} \int_{y=-2}^{2-x^2} (x+1) \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_{x=-2}^{2} (x+1)(4-x^2) \, \mathrm{d}x \\
&= \int_{x=-2}^{2} (4 + 4x - x^2 - x^3) \, \mathrm{d}x \\
&= \int_{x=-2}^{2} (4 - x^2) \, \mathrm{d}x && (4x \text{ and } -x^3 \text{ are odd functions}) \\
&= 2 \int_{x=0}^{2} (4 - x^2) \, \mathrm{d}x && (4 \text{ and } -x^2 \text{ are even functions}) \\
&= 2 \left(4x - \frac{x^3}{3} \right) \Big|_{x=0}^{2} = 2 \left(8 - \frac{8}{3} \right) = \frac{32}{3}.
\end{aligned}$$

2.2 Green's theorem for differential forms

From the point of view of differential forms, it is not surprising that Green's theorem applies to both flux and circulation: it is really the same theorem in both cases.

The vector field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ corresponds to the 1-form $M \, \mathrm{d}x + N \, \mathrm{d}y$. The common generalization of the circulation density $\operatorname{curl} \mathbf{F}$ and flux density $\operatorname{div} \mathbf{F}$, in this two-dimensional case, is the exterior derivative $\mathrm{d}(M \, \mathrm{d}x + N \, \mathrm{d}y)$. We have computed it before:

$$\mathrm{d}(M \, \mathrm{d}x + N \, \mathrm{d}y) = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y.$$

In the language of differential forms, the equation of Green's theorem simply states that if $\phi = M\,\mathrm{d}x + N\,\mathrm{d}y$, then

$$\int_C \phi = \iint_R \mathrm{d}\phi.$$

This is a very nice, compact expression, though the calculations are ultimately the same.

One difference here is the matter of orientation.

- For the vector field version of Green's theorem to work, the boundary C must be positively oriented: oriented counterclockwise. This means that our line integral must be computing the counterclockwise circulation around C, or the outward flux across C.
- For the differential form version of Green's theorem to work, there is no such requirement. Rather, the boundary of C must have a **compatible orientation** with the orientation of R, and we take an *oriented* integral over R.

What does "compatible orientation" mean? Well, if you go back to Lecture 3, you will see that an oriented region R in two dimensions is one that has a notion of "clockwise" and "counterclockwise" at each point. For the orientations of R and C to be compatible, the orientation of C must be *whatever the region R thinks is counterclockwise.*

A different way to put it: an oriented region R has a preferred direction of rotation at each point which it calls the positive direction. An oriented curve C has a preferred direction of motion at each point which it calls the positive direction. These two must match! (If you draw a tiny loop inside R oriented in R's positive direction, then near the boundary of R, it must be going the same way as the positive direction of C.)

How does this affect our calculations? Well, the oriented integral of $\mathrm{d}\phi$ over R will be reduced to an unoriented integral in one of two ways.

Case 1: If the rotation from positive x to positive y is the positive direction according to R, then we want to write $\mathrm{d}\phi$ as a multiple of $\mathrm{d}x \wedge \mathrm{d}y$. In this case,

$$\iint_R \mathrm{d}\phi = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y$$

where the last integral is the unoriented integral to take.

Case 2: If instead the rotation from positive y to positive x is the positive direction according to R, then we want to write $\mathrm{d}\phi$ as a multiple of $\mathrm{d}y \wedge \mathrm{d}x$. In this case,

$$\iint_R \mathrm{d}\phi = \iint_R \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \mathrm{d}y \wedge \mathrm{d}x = \iint_R \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \mathrm{d}y\,\mathrm{d}x$$

where the last integral is the unoriented integral to take.

If C is oriented counterclockwise, then we want R's positive direction to also be counterclockwise, which puts us in Case 1, and the unoriented integral we get is the one in the vector field version of Green's theorem.

If C is oriented clockwise, then we end up in Case 2, and the integral we get changes sign—but this is correct, because the clockwise line integral around C also has the opposite sign.

Why go to all this trouble? It's true that by remembering the rule "Green's theorem is only for counterclockwise circulation and outward flux", we avoid having to think about oriented regions. This is true… for now.

Once we get to surfaces in $\mathbb{R}^3$, and their boundaries, thinking about giving them compatible orientations will be unavoidable! So we should take some baby steps in that direction now, to make the transition less painful.