Math 3204: Calculus IV[1] Mikhail Lavrov

Lecture 15: Examples and applications of Green's theorem

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1 Area bounded by a curve

Though we often apply Green's theorem to turn a complicated line integral into a simple double integral, we can sometimes go in the other direction.

The most common example of this is for computing area. The area of a region R is given by the double integral $\iint_R \mathrm{d}A$. But if R is difficult to integrate over, and its boundary is easy to integrate over, then we might want to use Green's theorem.

Okay, but how? In order for Green's theorem to apply, we need a vector field $\mathbf{F}$ such that $\operatorname{curl} \mathbf{F} = 1$. Then, if C is the counterclockwise boundary of R, we conclude that

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F} \, \mathrm{d}A = \iint_R \mathrm{d}A,$$

which is the area of R.

There are many choices for $\mathbf{F}$. A simple one to use is the vector field $x\,\mathbf{j}$ (that is, $0\,\mathbf{i} + x\,\mathbf{j}$). It is not unique even among linear vector fields: $-y\,\mathbf{i}$, or $\frac{-y\,\mathbf{i}+x\,\mathbf{j}}{2}$, or $ay\,\mathbf{i} + bx\,\mathbf{j}$ for any a, b with $b - a = 1$, will all work. And we can further add on any conservative vector field of our choice: for example, we can take the vector field $\cos y\,\mathbf{i} + x(1 - \sin y)\,\mathbf{j}$, which adds $x\,\mathbf{j}$ to the gradient field of $f(x, y) = x \cos y$.

However, part of our plan is to take a line integral of $\mathbf{F}$ around C, so usually something simple is the way to go.

For our first example, let's find a formula for the area enclosed by the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

There are many approaches to finding a formula like this, including linear algebra and uv-substitutions. But we'll use Green's theorem.

The boundary of the ellipse can be given the counterclockwise parameterization

$$\mathbf{r}(t) = (a \cos t, b \sin t), \qquad t \in [0, 2\pi].$$

We will choose $\mathbf{F} = -\frac{1}{2}y\,\mathbf{i} + \frac{1}{2}x\,\mathbf{j}$, because I happen to know ahead of time that this will give us a nice integral.

[1]This document comes from an archive of the Math 3204 course webpage: http://misha.fish/archive/3204-fall-2024

As usual, we compute $\mathbf{F}(\mathbf{r}(t)) = -\frac{1}{2}b\sin t\,\mathbf{i} + \frac{1}{2}a\cos t\,\mathbf{j}$, $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = (-a\sin t, b\cos t)$, and then take their dot product, getting

$$\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{1}{2}ab\sin^2 t + \frac{1}{2}ab\cos^2 t = \frac{1}{2}ab.$$

So the area of the ellipse is

$$\int_{t=0}^{2\pi} \frac{1}{2}ab\,\mathrm{d}t = \frac{1}{2}ab \cdot 2\pi = \pi ab.$$

Here's a second neat application. Suppose that we have a k-sided polygon whose corners have coordinates $(x_1, y_1), (x_2, y_2), \dots, (x_k, y_k)$, in counterclockwise order around the polygon. We can use this to compute the area!

Once again, we will use $\mathbf{F} = -\frac{1}{2}y\,\mathbf{i} + \frac{1}{2}x\,\mathbf{j}$. Actually, to make life easier, let's use $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}$ (which has $\operatorname{curl}\mathbf{F} = 2$) and divide by 2 at the end. We parameterize the segment from (x_i, y_i) to (x_{i+1}, y_{i+1}) by

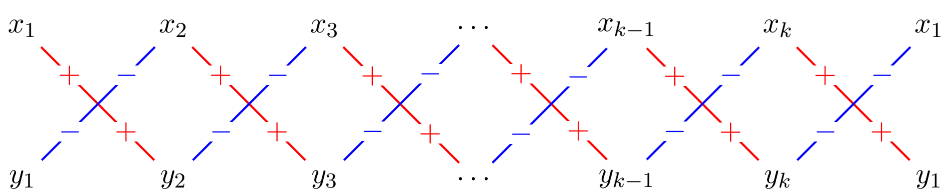
$$\mathbf{r}(t) = ((1-t)x_i + tx_{i+1}, (1-t)y_i + ty_{i+1}), \qquad t \in [0, 1].$$

We get $\mathbf{F}(\mathbf{r}(t)) = (-y_i + ty_i - ty_{i+1})\,\mathbf{i} + (x_i - tx_i + tx_{i+1})\,\mathbf{j}$ and $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = (x_{i+1} - x_i)\,\mathbf{i} + (y_{i+1} - y_i)\,\mathbf{j}$. When we take the dot product, many terms cancel: for example, $(ty_i)(-x_i)$ from the $\mathbf{i}$-components cancels with $(-tx_i)(-y_i)$ from the $\mathbf{j}$-components. In fact, the only terms that don't cancel are the ones that don't depend on t: the product simplifies to $x_i y_{i+1} - x_{i+1} y_i$.

Integrating this as t goes from 0 to 1 does nothing. Adding this up over all line segments, then dividing by 2, gives us the formula

$$A = \frac{x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + \cdots + x_{k-1}y_k - x_ky_{k-1} + x_ky_1 - x_1y_k}{2}.$$

This is called the "shoelace formula", from a diagram that represents the products we take:



The red lines with $+$'s on them are the products that have a positive sign; the blue lines with $-$'s on them are the products that have a negative sign; together, the lines resemble a shoelace pattern.

2 Regions with holes in them

Let R be the region bounded between the circle of radius 1 and the circle of radius 2 centered at $(0, 0)$; in other words,

$$R = \{(x, y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 4\}.$$

Let $\mathbf{F} = x^3\,\mathbf{i} + y^3\,\mathbf{j}$, and for the sake of variety, let's look at the flux and divergence version of Green's theorem.

We can integrate

$$\iint_R \operatorname{div} \mathbf{F} \, \mathrm{d}A = \iint_R 3(x^2 + y^2) \, \mathrm{d}A$$

by rewriting the integral in polar coordinates: replacing $3(x^2 + y^2)$ by $3r^2$, $\mathrm{d}A$ by $r \, \mathrm{d}r \, \mathrm{d}\theta$, and the bounds on our region by $1 \le r \le 2$ and $0 \le \theta \le 2\pi$. The result is

$$\int_{\theta=0}^{2\pi} \int_{r=1}^{2} 3r^3 \, \mathrm{d}r \, \mathrm{d}\theta = 2\pi \cdot \left. \frac{3r^4}{4} \right|_{r=1}^{2} = 2\pi \left(\frac{3 \cdot 16}{4} - \frac{3 \cdot 1}{4} \right) = \frac{45\pi}{2}.$$

What does this integral represent. Green's theorem does not directly apply to this region, because it is not simply connected. Nevertheless, the double integral of $\operatorname{div} \mathbf{F}$ over R *does* give us the outward flux across the boundary of R—just in a more complicated way.

The region R has two boundaries: the circle C_1 of radius 1, and the circle C_2 of radius 2. If we want to look at the flux out of R, then this will be the combination of the outward flux integral across C_2 and the *inward* flux integral across C_1: if we leave the region R by crossing C_1, this is C_1's inward direction! This suggests that

$$\iint_R \operatorname{div} \mathbf{F} \, \mathrm{d}A = \int_{C_2} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s - \int_{C_1} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s. \tag{1}$$

Here, we continue to orient both C_1 and C_2 counterclockwise, and follow all sign conventions for flux, but we subtract the integral over C_1 rather than adding it to obtain an inward flux.

Equation (1) is, indeed, true, and there are two ways to prove it from Green's theorem.

- Divide the region R in half by cutting it, for example, along the line $x = 0$. Let R' be the top half and R'' be the bottom half; let C' and C'' be the counterclockwise boundaries of R' and R'', respectively.

  Then by additivity of integrals and Green's theorem,

  $$\begin{aligned} \iint_R \operatorname{div} \mathbf{F} \, \mathrm{d}A &= \iint_{R'} \operatorname{div} \mathbf{F} \, \mathrm{d}A + \iint_{R''} \operatorname{div} \mathbf{F} \, \mathrm{d}A \\ &= \int_{C'} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s + \int_{C''} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s. \end{aligned}$$

  The curves C' and C'' each trace half of C_1 clockwise and each trace half of C_2 counterclockwise, which gives us the two flux integrals in (1). Additionally, both C' and C'' trace the line from $(-2, 0)$ to $(-1, 0)$ and from $(1, 0)$ to $(2, 0)$, but they do so in opposite directions—which makes sense, because when R' and R'' touch, flux out of R' is flux into R'', and vice versa. So those portions of the flux integrals cancel, and we get nothing other than (1).

- Let $R_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$ and let $R_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 4\}$: the boundary of R_1 is exactly C_1 and the boundary of R_2 is exactly C_2. Then R_2 is the union of two non-overlapping regions R_1 and R_2, so

  $$\iint_{R_2} \operatorname{div} \mathbf{F} \, \mathrm{d}A = \iint_{R_1} \operatorname{div} \mathbf{F} \, \mathrm{d}A + \iint_R \operatorname{div} \mathbf{F} \, \mathrm{d}A.$$

Even if we cannot apply Green's theorem to R, we can apply it to R_1 and R_2, getting

$$\int_{C_2} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s = \iint_{C_1} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s + \iint_R \operatorname{div} \mathbf{F} \, \mathrm{d}A$$

and this can be rearranged to get (1).

The second argument seems simpler, but it has a drawback: unlike the first argument, it requires $\mathbf{F}$ to be defined and differentiable throughout all of R_1, as well. One of the main reasons we might want to leave a hole in our region is to avoid places where a vector field $\mathbf{F}$ is undefined! It's a good thing we have the first argument, which is totally fine with that.

The intuition is clearer for flux, since "inward" and "outward" of region R have clear meanings, but a similar result holds for the circulation as well. Working with the same regions, we have

$$\iint_R \operatorname{curl} \mathbf{F} \, \mathrm{d}A = \int_{C_2} \mathbf{F} \cdot \mathbf{r} - \int_{C_1} \mathbf{F} \cdot \mathbf{r}. \tag{2}$$

Or, in other words, the integral of $\operatorname{curl} \mathbf{F}$ over R is equal to the sum of the counterclockwise circulation around C_2 and the *clockwise* circulation around C_1.

These facts generalize. Whenever we have a region R with one or more holes in it, we can apply Green's theorem to R anyway, but we must orient the boundary of R properly. The orientation of the boundary compatible with a counterclockwise orientation of R is the orientation that makes the "outside" boundary of R counterclockwise, and the "inside" boundaries clockwise.

Let's see an example of using this in a setting where we're forced to put a hole in R to avoid $\mathbf{F}$ being undefined. We will return to the gravity vector field

$$\mathbf{F} = -\frac{x\,\mathbf{i} + y\,\mathbf{j}}{(x^2 + y^2)^{3/2}}$$

and prove once again[2] that for any simple closed curve C not passing through the origin, $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$.

This is immediate if C does not contain $(0,0)$ inside it, since then we can apply Green's theorem to the interior of C, and $\operatorname{curl} \mathbf{F} = 0$. So suppose that C loops around the origin. Let C' be a circle of a small radius $\varepsilon > 0$ around $(0,0)$. Provided ε is small enough, C' is entirely contained inside C. If we let R be the region bounded between C' and C, then by (2), we have

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_{C'} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F} \, \mathrm{d}A = 0.$$

However, no matter what C is, we can compute the circulation integral around C'. To do this, parameterize it by

$$\mathbf{r}(t) = (\varepsilon \cos t, \varepsilon \sin t), \qquad t \in [0, 2\pi].$$

We have $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = -\varepsilon \sin t\,\mathbf{i} + \varepsilon \cos t\,\mathbf{j}$, and there is lots of cancellation in when we evaluate $\mathbf{F}(\mathbf{r}(t))$: $x^2 + y^2$ simplifies to ε^2, so $(x^2 + y^2)^{3/2}$ simplifies to ε^3, and we get

$$\mathbf{F}(\mathbf{r}(t)) = -\frac{\cos t\,\mathbf{i} + \sin t\,\mathbf{j}}{\varepsilon^2}.$$

[2] I apologize for the boring setting, but I wanted something where the algebra would not overwhelm us, and it's not like we haven't done anything interesting in this lecture.

The constant $\frac{1}{\varepsilon^2}$ looks dangerous, because it could be very very large—but in fact, $\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = (-\cos t)(-\sin t) + (-\sin t)(\cos t) = 0$, which stays 0 when we divide it by ε^2. So we integrate 0 from 0 to 2π, and get 0, and conclude that $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ as well.

There is another way to make this work out. Instead of making C' be a very tiny curve that's completely inside C, make C' be a very large curve that completely encloses C: parameterize it by

$$\mathbf{r}(t) = (A \cos t, A \sin t), \qquad t \in [0, 2\pi].$$

Here, we don't even have to do the work! All we need to know is that $\|\mathbf{F}\| = \frac{1}{x^2+y^2}$ at a point (x, y), so when we integrate $\mathbf{F} \cdot \mathbf{T}$ around a circle of radius A, we are integrating a quantity that's at most $\frac{1}{A^2}$ (in absolute value) over a curve of length $2\pi A$. Therefore the answer must be less than $\frac{2\pi}{A}$ in absolute value, and this is true for any constant $A > 0$.

What we conclude from this is that for all $A > 0$,

$$\left| \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} \right| = \left| \int_{C'} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \iint_R \operatorname{curl} \mathbf{F} \, \mathrm{d}A \right| \le \left| \frac{2\pi}{A} + 0 \right| = \frac{2\pi}{A}.$$

But we can make the right-hand side arbitrarily small, by making A bigger! So in fact, the circulation integral around C can only be 0.