

## Lecture 16: Parameterizing surfaces

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# 1 What is a parameterization of a surface?

## 1.1 Defining parameterizations

When we were dealing with curves in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , we gave them parameterizations: functions  $\mathbf{r}: [a, b] \rightarrow \mathbb{R}^2$  and  $\mathbf{r}: [a, b] \rightarrow \mathbb{R}^3$ . We are about to begin the study of surfaces in  $\mathbb{R}^3$ , and we will want to give them parameterizations, too. What should that mean?

At the very least, it is certain that we will want a parameterization  $\mathbf{r}$  to be a function  $D \rightarrow \mathbb{R}^3$ , where  $D$  is some subset of the plane. The surface  $S$  should be the image of  $\mathbf{r}$ : the set

$$S = \{\mathbf{r}(u, v) : (u, v) \in D\}.$$

Not all functions are equally good, though. Lets consider some possibilities.

The first question is the domain. Suppose that we want to describe the top half of a unit sphere: the set of points  $(x, y, z)$  with  $x^2 + y^2 + z^2 = 1$  and  $z \geq 0$ . There are two approaches:

- One is to solve for  $z$ , and set  $z = \sqrt{1 - x^2 - y^2}$ . Then, we can take the parameterization

$$\mathbf{r}(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$$

whose domain  $D$  is a disk: the set of all points  $(u, v) \in \mathbb{R}^2$  with  $u^2 + v^2 \leq 1$ .

- In spherical coordinates  $(\rho, \theta, \phi)$ , this hemisphere is described by  $\rho = 1$ ,  $0 \leq \theta \leq 2\pi$ , and  $0 \leq \phi \leq \pi/2$ . Converting back to rectangular coordinates, we get a parameterization in terms of  $\phi$  and  $\theta$ :

$$\mathbf{r}(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$$

where  $(\phi, \theta)$  lies in the rectangular region  $[0, \pi/2] \times [0, 2\pi]$ . (We can also, if we like, rename  $\phi$  and  $\theta$  to  $u$  and  $v$ .)

The second parameterization is better for several reasons. First of all, we will eventually want to integrate something over the domain of the parameterization, and writing an integral

$$\int_{\phi=0}^{\pi/2} \int_{\theta=0}^{2\pi} \dots d\theta d\phi$$

is much better than writing an integral over the disk  $D$ . Second, it generalizes better to other partial spheres. Finally, the rectangular bounds make it easier to describe the boundary of the surface, which will also be important to us later.

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<sup>1</sup>This document comes from an archive of the Math 3204 course webpage: <http://misha.fish/archive/3204-fall-2024>

For this reason, we make a distinction between two types of parameterizations. We will say that a function  $\mathbf{r}: [a, b] \times [c, d] \rightarrow \mathbb{R}^3$  is a **rectangular-domain** parameterization (provided that it satisfies some injectivity requirements we'll discuss in a moment). Meanwhile, a function  $\mathbf{r}: D \rightarrow \mathbb{R}^3$ , where  $D$  can be any subset of the plane, will be called a **wacky-domain** parameterization. (Needless to say, this terminology is not commonly accepted.)

Most of our theorems and formulas can be stated in general for wacky-domain parameterizations, so we will do that whenever possible. Aside from that, we will assume all our parameterizations are rectangular-domain parameterizations. (When I ask you to find a parameterization, please assume that I mean a rectangular-domain parameterization.) It will usually be best to use rectangular-domain parameterizations for evaluating surface integrals (unless we are in an exceptional case where some other domain makes life *much* simpler).

In either case, there are additional requirements on the behavior of the function  $\mathbf{r}$ . First of all,  $\mathbf{r}$  should be sufficiently nice to do calculus with; at the very least, its components should have continuous partial derivatives with respect to both variables. Second, it should be *mostly* injective.

I say “mostly” because if we look at our spherical parameterization  $\mathbf{r}(\phi, \theta)$ , we see several failures of injectivity. First of all, we have  $\mathbf{r}(\phi, 0) = \mathbf{r}(\phi, 2\pi)$  for any  $\phi$ . Even worse, when we set  $\phi = 0$ , then for all possible values of  $\theta$ ,  $\mathbf{r}(0, \theta)$  gives us the same point  $(0, 0, 1)$ . However, both of these things happen on the boundary of the rectangle  $[0, \pi/2] \times [0, 2\pi]$ , which is generally acceptable. It is a lot like a single-variable parameterization  $\mathbf{r}(t)$ , where if  $t$  ranges over  $[a, b]$ , we might see  $\mathbf{r}(a) = \mathbf{r}(b)$ .

If the parameterization fails to be injective somewhere other than the boundary of its domain, that could be worse. At best, it means our surface is self-intersecting, which can produce unintuitive results. At worst, it could mean that the parameterization folds back in on itself, representing a part of the surface multiple times; in that case, many of our formulas will just give wrong answers. Avoid this!

## 1.2 Ways to think about parameterizations

There are (at least) three intuitions you can have for what a parameterization does. See which of these resonate with you; if possible, try to think about it in multiple ways.

- Imagine the domain  $D$  of the parameterization  $\mathbf{r}: D \rightarrow \mathbb{R}^3$  as a piece of rubber. (Ideally, a rectangular piece of rubber!)

Then, the parameterization  $\mathbf{r}$  tells us how to bend and stretch the piece of rubber to fit it to the shape of the surface we want to understand.

- Imagine that the surface we are parameterizing is a world inhabited by people. Those people want to have coordinates for their surface to talk about where they are: these are pairs  $(u, v)$  that represent their location. (This is how latitude and longitude work for us: people living on the surface of a sphere.)

The parameterization  $\mathbf{r}$  is the function that interprets those coordinates: given a pair  $(u, v)$ , it outputs the point  $\mathbf{r}(u, v)$  on the surface which is represented by those coordinates.

- You have already spent some time working with parameterizations of curves  $\mathbf{r}: [a, b] \rightarrow \mathbb{R}^3$ .

We can leverage that understanding: a surface is just made up of many curves, and a surface parameterization is just made up of many curve parameterizations.

If  $\mathbf{r}: [a, b] \times [c, d] \rightarrow \mathbb{R}^3$  is a parameterization of a surface, then for every value of  $\mathbf{u} \in [a, b]$ , we can define a function  $\mathbf{r}_u: [c, d] \rightarrow \mathbb{R}^3$  by  $\mathbf{r}_u(v) = \mathbf{r}(u, v)$ . Each of these functions  $\mathbf{r}_u$  is the parameterization of a curve in  $\mathbb{R}^3$ . As  $u$  changes from  $a$  to  $b$ , the curve changes, and the changing curves sweep out a surface.

For example, given the parameterization of a hemisphere by  $\mathbf{r}(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$  where  $(\phi, \theta) \in [0, \pi/2] \times [0, 2\pi]$ , we can look at the curve parameterized by  $\mathbf{r}_\phi$  for a fixed  $\phi$ . This is a circle in the plane  $z = \cos \phi$  which is centered at  $(0, 0, \cos \phi)$  and has radius  $\sin \phi$ . As  $\phi$  changes from 0 to  $\pi/2$ , the circle expands and moves down, sweeping out the hemisphere.

We could also look at the curve parameterized by  $\mathbf{r}_\theta$  for a fixed  $\theta$ . This is an arc in the shape of a quarter-circle starting at  $(0, 0, 1)$  and ending at  $(\cos \theta, \sin \theta, 0)$ . As  $\theta$  changes from 0 to  $2\pi$ , the arc rotates around the  $z$ -axis, sweeping out the hemisphere in a different way.

These analogies are going to be helpful when coming up with parameterizations of tricky surfaces.

## 2 Examples of parameterizations

Let's parameterize some surfaces.

### 2.1 Surfaces above a rectangle

A good case to start with is a surface which is given by an equation  $z = f(x, y)$ , and is bounded by separate inequalities on  $x$  and  $y$ :  $a \leq x \leq b$  and  $c \leq y \leq d$ . (We can imagine the surface as lying above the rectangle  $[a, b] \times [c, d]$  in the  $xy$ -plane, but nothing changes if the surface is sometimes or always below that plane.)

In this case, our parameterization can look very simple: set  $x = u$  and  $y = v$ . We have

$$\mathbf{r}(u, v) = (u, v, f(u, v)), \quad (u, v) \in [a, b] \times [c, d].$$

There is nothing special about the variables  $x$  and  $y$ , of course. For example, suppose that we start with the cylinder  $x^2 + y^2 = 1$ , where  $0 \leq z \leq 1$ , and chop it in half, taking only the half with  $y \geq 0$  as well. Then this cylinder is a surface given by the equation  $y = \sqrt{1 - x^2}$ , where  $x$  is free to range from  $-1$  to  $1$ , and  $z$  still goes from 0 to 1. So if we set  $x = u$  and  $z = v$ , we get the parameterization

$$\mathbf{r}(u, v) = (u, \sqrt{1 - u^2}, v), \quad (u, v) \in [-1, 1] \times [0, 1].$$

(We would *not* be able to use this parameterization for the whole cylinder, because then, there would be multiple values of  $y$  for the same  $(x, z)$  pair.)

When the surface is given by an equation  $z = f(x, y)$ , and lies above a region  $D$  in the  $xy$ -plane that's *not* a rectangle, this could theoretically be a reason to use a wacky-domain parameterization. Emphasis on "could". You should stop to think if you have any alternatives.

One of the alternatives you might find is to use a different coordinate system for the region  $D$  which *makes* it rectangular. To give an example, let's take the surface  $z = x^2 - y^2$ , but bound  $x$  and  $y$  by the inequality  $|x| + |y| \leq 1$ , which forms a diamond in the  $xy$ -plane. It's true, we could parameterize this region by  $\mathbf{r}(u, v) = (u, v, u^2 - v^2)$ , where  $(u, v)$  are points in this diamond. But think about it; is there anything else we can do?

If we were taking a double integral over the diamond  $\{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$ , we might consider a  $uv$ -substitution: we'd set  $u = x + y$  and  $v = x - y$ , so that the sides of the diamond are now defined by the inequalities  $-1 \leq u \leq 1$  and  $-1 \leq v \leq 1$ . We can do the same thing here! Solving for  $x$  and  $y$ , we get  $x = \frac{u+v}{2}$  and  $y = \frac{u-v}{2}$ . Meanwhile,  $z = x^2 - y^2 = (x + y)(x - y)$  is just the product  $uv$ . So our surface has the much nicer parameterization

$$\mathbf{r}(u, v) = \left( \frac{u+v}{2}, \frac{u-v}{2}, uv \right), \quad (u, v) \in [-1, 1] \times [-1, 1].$$

While we're on the topic of substitutions, it often helps to check whether our surface can be described easily in cylindrical or spherical coordinates. An especially common scenario: if our surface has the equation  $z = f(x, y)$ , but the bounds on  $x$  and  $y$  give the shape of a *circle* in the  $xy$ -plane (or some other "round") shape, then maybe we can rewrite  $z$  in terms of  $r$  and  $\theta$  and set  $r = u$ ,  $\theta = v$ . In such cases, it's important not to forget that **the formula for  $\mathbf{r}(u, v)$  should always be in rectangular coordinates**. Cylindrical coordinates are just a source of inspiration!

For example, suppose we decide to take the infinite paraboloid  $z = x^2 + y^2$ , and cut it off by the inequality  $z \leq 3$ . That inequality is equivalent to the inequality  $x^2 + y^2 \leq 3$ , defining a circle of radius  $\sqrt{3}$  in the  $xy$ -plane. So we decide to switch to cylindrical coordinates.

In cylindrical coordinates,  $z = r^2$ ,  $r$  ranges from 0 to  $\sqrt{3}$ , and  $\theta$  ranges from 0 to  $2\pi$ . So we set  $r = u$  and  $\theta = v$ , and prepare to write down our parameterization. Using the coordinate change formulas,  $x = r \cos \theta = u \cos v$ , while  $y = r \sin \theta = u \sin v$ . Again,  $z = r^2 = u^2$ . So we get the parameterization

$$\mathbf{r}(u, v) = (u \cos v, u \sin v, u^2), \quad (u, v) \in [0, \sqrt{3}] \times [0, 2\pi].$$

## 2.2 Transformations

It might turn out that the tools we've discovered so far almost, but not quite, let us describe the surface we want. We just need to modify our surface slightly. Well, there are a few transformations of  $\mathbb{R}^3$  that are easy to describe with coordinates, so they're easy to apply to our parameterizations.

First, we can translate our surface by a vector. If we add constants  $a, b, c$  to the three components of  $\mathbf{r}(u, v)$ , that shifts every point of our surface by the vector  $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ . For example,

$$\mathbf{r}(u, v) = (u \cos v + 1, u \sin v + 2, u^2 + 3), \quad (u, v) \in [0, \sqrt{3}] \times [0, 2\pi]$$

defines another paraboloid congruent to the first, but its vertex is now at the point  $(1, 2, 3)$  instead of  $(0, 0, 0)$ .

We can scale the coordinates, together or independently of each other, by *multiplying* them by constants. For example,

$$\mathbf{r}(u, v) = (u \cos v, \tfrac{1}{2}u \sin v, u^2), \quad (u, v) \in [0, \sqrt{3}] \times [0, 2\pi]$$

takes the paraboloid and squishes it in the  $y$ -direction, making it have half the width in that direction. Usually, we want to scale before translating, because scaling fixes the point  $(0, 0, 0)$  in place, and we want to do this operation while  $(0, 0, 0)$  is still a meaningful point to keep fixed.

Rotations and reflections are another powerful tool. In previous lectures, we've already seen how some of these can be accomplished just by swapping around coordinates, and that continues to be true for surfaces. Other, more complicated rotations properly should be studied in a linear algebra class. For example, a rotation by  $45^\circ$  around the  $x$ -axis can be accomplished by the linear transformation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Apply this transformation to our paraboloid, and you'll get

$$\mathbf{r}(u, v) = \left( u \cos v, \frac{\sqrt{2}}{2} \sin v - \frac{\sqrt{2}}{2} u^2, \frac{\sqrt{2}}{2} \sin v + \frac{\sqrt{2}}{2} u^2 \right), \quad (u, v) \in [0, \sqrt{3}] \times [0, 2\pi].$$

This is still a paraboloid whose vertex is at  $(0, 0, 0)$ , but now it opens in the direction of the diagonal vector  $-\mathbf{j} + \mathbf{k}$  rather than the vertical vector  $\mathbf{k}$ .

## 2.3 A tricky example

If you take a long and narrow strip of paper, give it a half-twist, and then tape the two ends together, you get an unusual surface called a Möbius strip. This is a surface with some weird properties: if you imagine an ant crawling along the strip of paper, then once it makes a full loop, it will end up on the other side of the paper from where it started! In a sense, the Möbius strip is a “one-sided” surface.

That property of the Möbius strip will actually make it an important example for us to consider later, but for now, let's just give it a parameterization.

As our prototype, let's take a “non-Möbius” strip: a loop of paper that doesn't twist at all. This is essentially the curved boundary of a wide and short cylinder. Let's say that the cylinder has radius 5, and ranges vertically from  $z = 1$  to  $z = -1$ ; I've made the range of  $z$  symmetric, because the “center” of the strip, where  $z = 0$ , will be useful to us later. The parameterization now is

$$\mathbf{r}(u, v) = (5 \cos u, 5 \sin u, v), \quad (u, v) \in [0, 2\pi] \times [-1, 1].$$

I think a good way to visualize how we want to twist the strip of paper is to imagine it as a surface swept out by the curve  $\mathbf{r}_u(v)$  as  $u$  varies from 0 to  $2\pi$ . For a fixed  $u$ , this is just a segment of length 2 from one side of the strip to the other: from  $(5 \cos u, 5 \sin u, -1)$  to  $(5 \cos u, 5 \sin u, 1)$ .

To make this segment “twist” as we go, we should have it rotate about its center. This is easiest to visualize in cylindrical coordinates, in the  $rz$ -half-plane. In that half-plane, our untwisted segment starts from  $(r, z) = (5, -1)$ , and goes from  $(r, z) = (5, 1)$ .

To rotate it by an angle  $\alpha$  about the point  $(r, z) = (5, 0)$ , we can write the starting point as  $(r, z) = (5 + \sin \alpha, -\cos \alpha)$  and the ending point as  $(r, z) = (5 - \sin \alpha, \cos \alpha)$ . The entire segment would then have  $r = 5 - v \sin \alpha$  and  $z = v \cos \alpha$ .

How should  $\alpha$  depend on the angle  $u$  (in other words, on  $\theta$ )? It is tempting to set  $\alpha = u$ . But in that case, the segment will make a full  $360^\circ$  rotation about its center just as it makes a full  $360^\circ$  revolution about the  $z$ -axis. That would be a full twist, joining up the same side of the strip of paper. To get a Möbius strip, we want it to rotate half as fast, so we set  $\alpha = u/2$ .

We now know our cylindrical coordinates: they are

$$\begin{aligned} r &= 5 - v \sin \frac{u}{2} \\ z &= v \cos \frac{u}{2} \\ \theta &= u \end{aligned}$$

where  $u \in [0, 2\pi]$  and  $v \in [-1, 1]$ . To get our parameterization, we convert back to rectangular coordinates, with  $x = r \cos \theta$  and  $y = r \sin \theta$ . Our final answer is

$$\mathbf{r}(u, v) = \left( \left( 5 - v \sin \frac{u}{2} \right) \cos u, \left( 5 - v \sin \frac{u}{2} \right) \sin u, v \cos \frac{u}{2} \right), \quad (u, v) \in [0, 2\pi] \times [-1, 1].$$

Here is the surface we've described:

