Lecture 17: Surface area

1 Two approaches to area

Our goal today is to learn how to use integrals to find surface areas. In order to do that, we have to solve a fundamental sub-problem: given two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$, what is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$? (That is, the parallelogram with corners $\mathbf{0}$, $\mathbf{a}$, $\mathbf{a}+\mathbf{b}$, and $\mathbf{b}$.)

As a special case, suppose that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal. Then the parallelogram they form is actually a rectangle, and we can find its area in a straightforward fashion: it is simply the product $\|\mathbf{a}\| \cdot \|\mathbf{b}\|$. Keep this formula in mind: it can often be a useful shortcut!

In general, the area is $\|\mathbf{a}\|\|\mathbf{b}\| \sin\theta$, where θ is the angle formed between $\mathbf{a}$ and $\mathbf{b}$, but computing that angle is actually kind of annoying. We'd rather have an algebraic formula for area that uses only the components of $\mathbf{a}$ and $\mathbf{b}$.

It turns out that there are two ways to do this. They give the same answer, of course, but the formula will look very different. (Personally, I think the Gram matrix method is simpler, but cross products might be familiar to some of you already—and we'll need them later anyway.)

1.1 Cross products

Given vectors $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ and $\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$, we define the cross product $\mathbf{a}\times\mathbf{b}$ by the formula

$$\mathbf{a}\times\mathbf{b} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = (a_2b_3 - a_3b_2)\,\mathbf{i} + (a_3b_1 - a_1b_3)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k}.$$

The cross product of $\mathbf{a}$ and $\mathbf{b}$ has several important geometric properties.

1. Most importantly for us, the magnitude $\|\mathbf{a}\times\mathbf{b}\|$ is the area of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$.
2. This follows from a more general fact: the absolute value $|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|$ is the volume of the parallelepiped (a 3D analog of a parallelogram) spanned by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. That's because taking the dot product with $\mathbf{c} = c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}$ is equivalent to replacing $\mathbf{i}, \mathbf{j}, \mathbf{k}$ by c_1, c_2, c_3 in the formula, which gives the determinant of a 3×3 matrix with rows $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.
3. In particular, $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = 0$ (they're the volume of a flat parallelepiped), so $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

[1]This document comes from an archive of the Math 3204 course webpage: http://misha.fish/archive/3204-fall-2024

The first property tells us the magnitude of $\mathbf{a}\times\mathbf{b}$, and the third property *almost* tells us the direction. There is one ambiguity: if $\mathbf{c}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, then $-\mathbf{c}$ is another orthogonal vector with the same magnitude.

The determinant formula for $\mathbf{a} \times \mathbf{b}$ picks the vector that follows the **right-hand rule**:

> "If you place your right hand on the **ab**-plane in the direction of **a** with your fingers curling toward **b**, then your thumb will point in the direction of $\mathbf{a} \times \mathbf{b}$."

This can require great physical contortion for some cross products; it's best if you're good at imagining your hand in various positions without actually moving it.

As a corollary of this rule, or the determinant formula, the cross product is **anti-commutative**: $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$.

1.2 The Gram matrix

The formula $\|\mathbf{a} \times \mathbf{b}\|$ for area is a bit arbitrary: it's something that works in three dimensions but not in any other number. Here is a formula that could be used for areas in any number of dimensions!

The **Gram matrix** of $\mathbf{a}$ and $\mathbf{b}$ is the "multiplication table of their dot products": the matrix

$$G = \begin{bmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} \end{bmatrix}$$

Because dot products are commutative, this matrix is symmetric: we only need to evaluate three dot products to compute it.

The area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ is now given by $\sqrt{\det G}$: the square root of the determinant of the Gram matrix (which will always be positive).

Here's why the formula works. The area we want is $\|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$; however, we already know that $\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$. So we can write G as

$$G = \begin{bmatrix} \|\mathbf{a}\|^2 & \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta \\ \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta & \|\mathbf{b}\|^2 \end{bmatrix}.$$

When we compute the determinant, we'll get $\|\mathbf{a}\|^2\|\mathbf{b}\|^2 - \|\mathbf{a}\|^2\|\mathbf{b}\|^2\cos^2\theta$, which simplifies to $\|\mathbf{a}\|^2\|\mathbf{b}\|^2\sin^2\theta$ because $1 - \cos^2\theta = \sin^2\theta$. As a result, when we take the square root, we get the area we wanted.

It is less obvious—but no less true—that a further generalization of the Gram matrix method also works. Suppose that you are working in $\mathbb{R}^n$, and want to know the k-dimensional volume of a "parallelotope" spanned by k vectors. We can construct a $k \times k$ Gram matrix whose entries are the possible dot products of the vectors, and $\sqrt{\det G}$ will still give us the volume we want!

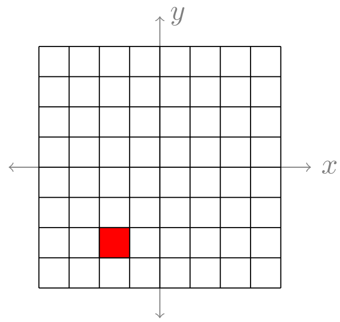
2 Surface area

Whichever formula we decide to use, we will need it to derive a formula for our first surface integral: the surface area integral.

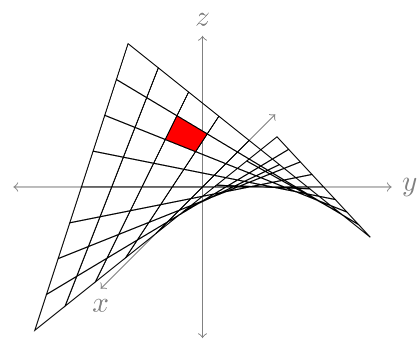
As our example, let's take the surface with equation $z = xy$, with $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. (A rotated version of this surface appeared in an example in the previous lecture.) We can set $x = u$ and $y = v$ to parameterize this surface, getting

$$\mathbf{r}(u, v) = (u, v, uv) \qquad (u, v) \in [-1, 1] \times [-1, 1].$$

You can see a picture of this surface in Figure 1b; it is called a "hyperboloid" and is notable for the saddle-shaped bend it has near the origin.



(a) The domain of the parameterization



(b) The surface $z = xy$

Figure 1: Visualizing the scalar line integral of a 2-variable function

To find the area of a surface, we will start with a discrete approximation and then take a limit, as usual.

If our parameterization has domain $[a, b] \times [c, d]$ in the uv-plane, we can divide this domain into many square (or rectangular) cells, as shown in Figure 1a. Each of these square cells is mapped by $\mathbf{r}$ to a tiny piece of the surface; the cells highlighted in red in Figure 1 are an example of this. Even this tiny piece of the surface has some curvature, but as an approximation, we will pretend that it is flat.

If the cell in the uv-plane is $[u, u + \Delta u] \times [v, v + \Delta v]$, then $\mathbf{r}$ sends that cell to an approximate parallelogram in $\mathbb{R}^3$, with one side parallel to $\mathbf{a} = \mathbf{r}(u + \Delta u, v) - \mathbf{r}(u, v)$ and one side parallel to $\mathbf{b} = \mathbf{r}(u, v + \Delta v) - \mathbf{r}(u, v)$. The area of that parallelogram is $\|\mathbf{a} \times \mathbf{b}\|$, which we can rewrite as

$$\left\| \frac{\mathbf{r}(u + \Delta u, v) - \mathbf{r}(u, v)}{\Delta u} \times \frac{\mathbf{r}(u, v + \Delta v) - \mathbf{r}(u, v)}{\Delta v} \right\| \Delta u \, \Delta v. \tag{1}$$

(A fact we're implicitly using here is that the cross product is linear: for a constant $k \in \mathbb{R}$, $(k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b}) = k(\mathbf{a} \times \mathbf{b})$. This is true because the determinant is linear.)

In the limit as $\Delta u \to 0$ and $\Delta v \to 0$, the two inputs to the cross product of (1) approach the partial derivatives $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$. If we sum this expression over all the small cells making up the domain $[a, b] \times [c, d]$, we get an integral

$$\int_{u=a}^{b} \int_{v=c}^{d} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \mathrm{d}v \, \mathrm{d}u.$$

This is the integral we will use to find surface areas!

Instead of $\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\|$, we can also use $\sqrt{\det G}$, where G is the Gram matrix of $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$.

In our example, $\mathbf{r}(u,v) = (u, v, uv)$, so $\frac{\partial \mathbf{r}}{\partial u} = \mathbf{i} + v\,\mathbf{k}$ and $\frac{\partial \mathbf{r}}{\partial v} = \mathbf{j} + u\,\mathbf{k}$. Their cross product is

$$(\mathbf{i} + v\,\mathbf{k}) \times (\mathbf{j} + u\,\mathbf{k}) = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & v \\ 0 & 1 & u \end{bmatrix} = -v\,\mathbf{i} - u\,\mathbf{j} + \mathbf{k}.$$

Therefore $\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\| = \sqrt{(-v)^2 + (-u)^2 + 1} = \sqrt{1 + u^2 + v^2}$, and the surface area of the hyperboloid is given by

$$\int_{u=-1}^{1} \int_{v=-1}^{1} \sqrt{1 + u^2 + v^2}\,\mathrm{d}v\,\mathrm{d}u.$$

Let's try the Gram matrix method as well. We already know $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$. To find the entries of the Gram matrix, we compute $\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial u} = 1 + v^2$, $\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} = uv$, and $\frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial v} = 1 + u^2$. Therefore

$$G = \begin{bmatrix} 1 + v^2 & uv \\ uv & 1 + u^2 \end{bmatrix}$$

and $\det G = (1 + v^2)(1 + u^2) - (uv)^2 = 1 + u^2 + v^2$. Once again, we get $\sqrt{1 + u^2 + v^2}\,\mathrm{d}v\,\mathrm{d}u$ as the area element in our integral.

No matter how we obtain it, the integral

$$\int_{u=-1}^{1} \int_{v=-1}^{1} \sqrt{1 + u^2 + v^2}\,\mathrm{d}v\,\mathrm{d}u.$$

simplifies to $\frac{4}{3}(\sqrt{3} + \ln(7 + 4\sqrt{3})) - \frac{2}{9}\pi$ after only a couple of terrifying trig substitutions that I will not include here.

3 The trick to spherical surface area

Let's use this technique to find the surface area of a sphere. Center our sphere at $(0,0,0)$ and give it radius ρ. We can give it a parameterization based on spherical coordinates:

$$\mathbf{r}(\phi, \theta) = (\rho \cos\theta \sin\phi, \rho \sin\theta \sin\phi, \rho \cos\phi), \qquad (\phi, \theta) \in [0, \pi] \times [0, 2\pi].$$

(Though ρ here is just the spherical coordinate ρ, in this parameterization it is held constant; it is not one of the variables.)

The partial derivatives here are

$$\frac{\partial \mathbf{r}}{\partial \phi} = \rho \cos\theta \cos\phi\,\mathbf{i} + \rho \sin\theta \cos\phi\,\mathbf{j} - \rho \sin\phi\,\mathbf{k}$$

and

$$\frac{\partial \mathbf{r}}{\partial \theta} = -\rho \sin\theta \sin\phi\,\mathbf{i} + \rho \cos\theta \sin\phi\,\mathbf{j}.$$

It looks like we're in for a slog!

Things are not so bad, however. We can compute $\frac{\partial \mathbf{r}}{\partial \phi} \cdot \frac{\partial \mathbf{r}}{\partial \theta}$ with minimal suffering: both the **i**-components and the **j**-components give us $\rho^2 \sin\theta\cos\theta\sin\phi\cos\phi$ in the product (just in a different order), but they have opposite signs. So the dot product is 0: the two vectors are orthogonal! Therefore the area element in our integral can just be

$$\left\|\frac{\partial \mathbf{r}}{\partial \phi}\right\| \cdot \left\|\frac{\partial \mathbf{r}}{\partial \theta}\right\| \,\mathrm{d}\phi\,\mathrm{d}\theta.$$

The norm of $\frac{\partial \mathbf{r}}{\partial \phi}$ is $\sqrt{\rho^2\cos^2\theta\cos^2\phi + \rho^2\sin^2\theta\sin^2\phi + \rho^2\sin^2\phi}$. The first two terms under the square root combine into a $\rho^2\cos^2\phi$, because $\cos^2\theta + \sin^2\theta = 1$. Then, $\rho^2\cos^2\phi$ combines with the last term $\rho^2\sin^2\phi$ into a ρ^2, giving us $\left\|\frac{\partial \mathbf{r}}{\partial \phi}\right\| = \rho$.

The norm of $\frac{\partial \mathbf{r}}{\partial \theta}$ is $\sqrt{\rho^2\sin^2\theta\sin^2\phi + \rho^2\cos^2\theta\sin^2\phi}$, which simplifies to $\sqrt{\rho^2\sin^2\phi}$, so $\left\|\frac{\partial \mathbf{r}}{\partial \theta}\right\| = \rho\sin\phi$.

The product of these two factors might be familiar! Just like we got $\mathrm{d}V = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$ for the volume element when integrating in spherical coordinates, so do we get $\mathrm{d}A = \rho^2\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta$ for the area element when integrating over the surface of a sphere of radius ρ. In particular,

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \rho^2\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta$$

is the integral that gives us the area of a sphere of radius ρ. Integrating $\sin\phi$ from 0 to π gives 2, which we multiply by ρ^2 and then by 2π (the length of the domain of θ), so the surface area of the sphere is $4\pi\rho^2$.

Should we have expected $\frac{\partial \mathbf{r}}{\partial \phi}$ and $\frac{\partial \mathbf{r}}{\partial \theta}$ to be orthogonal? Well, if you're fast at drawing connections, you might remember that when you look at a globe, the parallel and meridian lines always meet at right angles. These are equivalent facts! So it's not a coincidence that our parallelograms turn out to be rectangles here.

If we don't notice this, one nice feature of the Gram matrix method is that computing the dot product $\frac{\partial \mathbf{r}}{\partial \phi} \cdot \frac{\partial \mathbf{r}}{\partial \theta}$ is a key step in the process anyway! In fact, if you happen to use the Gram matrix to compute the area of the parallelogram spanned by **a** and **b**, and you happen to be dealing with orthogonal vectors, then the method gracefully transitions into the special case: we will get

$$\sqrt{\det G} = \sqrt{\det\begin{bmatrix}\mathbf{a}\cdot\mathbf{a} & 0 \\ 0 & \mathbf{b}\cdot\mathbf{b}\end{bmatrix}} = \sqrt{(\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - 0} = \sqrt{\mathbf{a}\cdot\mathbf{a}}\cdot\sqrt{\mathbf{b}\cdot\mathbf{b}} = \|\mathbf{a}\|\cdot\|\mathbf{b}\|.$$

Unfortunately, if we attempt to compute the cross product $\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta}$, it ends up rather painful, though trig identities ensure that the truth wins out in the end.

4 Optional: Gram matrices in higher dimensions

In this section, I will prove the generalization of the Gram matrix method to finding the volume of a k-dimensional parallelotope in n-dimensional space. We will not cover this in class for two reasons:

1. We will not need any other kind of Gram matrices for dealing with integrals in $\mathbb{R}^2$ and $\mathbb{R}^3$;
2. This proof requires more linear algebra background than this class typically assumes.

I will write $\mathrm{Vol}(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(k)})$, where $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)} \in \mathbb{R}^n$, for the k-dimensional volume of the parallelotope spanned by $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)}$.

As a starting point, if $k = n$, then let X be the $n \times n$ matrix whose columns are $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$. Here, we have

$$\mathrm{Vol}(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}) = |\det(X)|.$$

(More generally, $\det(X)$ is the *signed* volume, taking orientation into account.) This starting point is proven in a linear algebra class by giving some axioms for how signed volume should behave—it should be linear in its arguments, invariant under certain row operations, and it should be 1 for the standard basis vectors—and proving that the determinant is the only function that satisfies these axioms.

To look at lower-dimensional volumes, we need another relationship. Suppose that $\mathbf{x}^{(k)}$ is orthogonal to all the previous vectors $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k-1)}$. Then we have

$$\mathrm{Vol}(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k-1)}, \mathbf{x}^{(k)}) = \mathrm{Vol}(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k-1)}) \cdot \left\|\mathbf{x}^{(k)}\right\|.$$

The idea here is that the k-dimensional parallelotope is a right prism built on top of the $(k-1)$-dimensional parallelotope spanned by $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k-1)}$ with height $\|\mathbf{x}^{(k)}\|$. The right-hand side of the formula above generalizes the "base times height" formula of area and volume.

Iterating this argument, suppose that $\mathbf{x}^{(k+1)}, \dots, \mathbf{x}^{(n)}$ are all orthogonal to $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)}$ and to each other. Then we have

$$\mathrm{Vol}(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}) = \mathrm{Vol}(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)}) \left\|\mathbf{x}^{(k+1)}\right\| \cdots \left\|\mathbf{x}^{(n)}\right\|.$$

In particular, if $\mathbf{x}^{(k+1)}, \dots, \mathbf{x}^{(n)}$ are all unit vectors and have length 1, then

$$\mathrm{Vol}(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}) = \mathrm{Vol}(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)}).$$

This is promising: it relates the k-dimensional volume we want to find to an n-dimensional volume we already have a formula for.

If we start with only $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)}$, we can use the Gram–Schmidt process (same Gram!) to construct some vectors $\mathbf{x}^{(k+1)}, \dots, \mathbf{x}^{(n)}$ with the properties above. Then the volume we want is given by $|\det(X)|$, where X is once again the $n \times n$ matrix whose columns are $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$. Unfortunately, this is not a useful formula, because we don't know a nice algebraic expression for the "extra" vectors we added to make this work.

To fix this, observe that $\det(X)^2 = \det(X^\mathsf{T})\det(X) = \det(X^\mathsf{T}X)$. The matrix $X^\mathsf{T}X$ is the Gram matrix for all n vectors, by the definition of matrix multiplication: its (i, j) entry is given by the dot product $\mathbf{x}^{(i)} \cdot \mathbf{x}^{(j)}$. In particular, its top left $k \times k$ submatrix is G, the Gram matrix of $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)}$ alone. Because the added vectors $\mathbf{x}^{(k+1)}, \dots, \mathbf{x}^{(n)}$ are unit vectors, the rest of the main diagonal is 1; because each one is orthogonal to everything else, all other entries outside the top left $k \times k$ block are 0. In other words, $X^\mathsf{T} X$ has the block structure:

$$X^\mathsf{T} X = \begin{bmatrix} G & 0_{k \times n-k} \\ 0_{n-k \times k} & I \end{bmatrix}.$$

As a result, we have $\det(X^\mathsf{T} X) = \det(G)\det(I) = \det(G)$. We already know that $\det(X^\mathsf{T} X) = \det(X)^2 = \mathrm{Vol}(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)})^2$, and so if we take the square root, we learn that

$$\mathrm{Vol}(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)}) = \sqrt{\det(G)},$$

which is the formula we wanted.