Math 3204: Calculus IV[1] Mikhail Lavrov

Lecture 21: More vector surface integrals

October 24, 2024 *Kennesaw State University*

1 Surfaces defined implicitly

1.1 An example with differential forms

Let's take the sideways-pointing cone $y^2 = x^2 + z^2$ where $0 \le y \le 1$ and slice it in half, taking only the portion with $z \ge 0$. Then, take the flux integral of $\mathbf{F} = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$ across the resulting surface S.

The flux integral in which direction? Well, this is one of those situations where our surface has a one-to-one projection onto the xy-plane, so that we can describe it in terms of x and y. In that case, it makes sense to distinguish the normal vectors at a point between the *upward* and the *downward* normal vectors—though, of course, both of them will have horizontal components as well. For this example, let's take the upward flux.

We *could* solve for z in terms of x and y, getting $z = \sqrt{y^2 - x^2}$, and eventually we will need to do that. However, to begin with, let's avoid doing that. Instead, take the total derivative on our surface to get

$$2y\,\mathrm{d}y = 2x\,\mathrm{d}x + 2z\,\mathrm{d}z.$$

This is useful if we're going to use exterior products to write our flux integral:

$$\iint_S \mathrm{d}y \wedge \mathrm{d}z + 2\,\mathrm{d}z \wedge \mathrm{d}x + 3\,\mathrm{d}x \wedge \mathrm{d}y.$$

We want an integral over a region in the xy-plane, so we want to express everything in terms of x, y, $\mathrm{d}x$, and $\mathrm{d}y$. To begin with, we can replace $\mathrm{d}z$ by $\frac{2y\,\mathrm{d}y - 2x\,\mathrm{d}x}{2z} = \frac{y}{z}\,\mathrm{d}y - \frac{x}{z}\,\mathrm{d}x$. In particular, $\mathrm{d}y \wedge \mathrm{d}z = -\frac{x}{z}\,\mathrm{d}y \wedge \mathrm{d}x = \frac{x}{z}\,\mathrm{d}x \wedge \mathrm{d}y$, and $\mathrm{d}z \wedge \mathrm{d}x = \frac{y}{z}\,\mathrm{d}y \wedge \mathrm{d}x = -\frac{y}{z}\,\mathrm{d}x \wedge \mathrm{d}y$. So our integral simplifies to

$$\iint_S \left(\frac{x}{z} - \frac{y}{z} + 3\right) \mathrm{d}x \wedge \mathrm{d}y = \iint_S \left(\frac{x-y}{\sqrt{y^2 - x^2}} + 3\right) \mathrm{d}x \wedge \mathrm{d}y.$$

This is now entirely in terms of the right variables, but still a weird oriented surface integral. We do two things to change that. First: since we want an upward integral, and the upward normal vector in the xy-plane corresponds to the 2-form $\mathrm{d}x \wedge \mathrm{d}y$ by the right-hand rule. (For the downward integral, we'd want to use $\mathrm{d}y \wedge \mathrm{d}x$, instead, negating everything.) Second: if $y^2 = x^2 + z^2$, then $y^2 \ge x^2$, so our range on x is $-y \le x \le y$; we have already decided that $0 \le y \le 1$. This gives us the bounds on our integral. The final result is:

$$\int_{y=0}^{1} \int_{x=-y}^{y} \left(\frac{x-y}{\sqrt{y^2 - x^2}} + 3\right) \mathrm{d}x\,\mathrm{d}y.$$

[1]This document comes from an archive of the Math 3204 course webpage: http://misha.fish/archive/3204-fall-2024

We can't entirely neglect our skills of taking integrals, so let's take this one. In order from easiest to hardest term:

- Integrating 3 over these bounds gives us 3 times the area of the triangle bounded by $0 \le y \le 1$ and $-x \le y \le x$. That area is 1, so the integral is 3.
- Integrating $\frac{x}{\sqrt{y^2-x^2}}$ over bounds that are symmetric in x gives 0, because this is an odd function of x.
- Unfortunately, even though $\frac{-y}{\sqrt{y^2-x^2}}$ is an odd function of y, this doesn't help us, because the bounds on y are not symmetric about the origin. Instead, begin by using the fact that it's an *even* function of x to take a different integral:

$$\int_{y=0}^{1}\int_{x=-y}^{y}\frac{-y}{\sqrt{y^2-x^2}}\,\mathrm{d}x\,\mathrm{d}y = 2\int_{y=0}^{1}\int_{x=0}^{y}\frac{-y}{\sqrt{y^2-x^2}}\,\mathrm{d}x\,\mathrm{d}y$$
$$= 2\int_{x=0}^{1}\int_{y=x}^{1}\frac{-y}{\sqrt{y^2-x^2}}\,\mathrm{d}y\,\mathrm{d}x.$$

  Now the inside integral can be done with the u-substitution $u = y^2 - x^2$, with $\mathrm{d}u = 2y\,\mathrm{d}y$. If y goes from x to 1, then u goes from 0 to $1 - x^2$, and we get the integral

$$\int_{x=0}^{1}\int_{u=0}^{1-x^2}\frac{-1}{\sqrt{u}}\,\mathrm{d}u\,\mathrm{d}x = \int_{x=0}^{1} -2\sqrt{u}\Big|_{u=0}^{1-x^2}\,\mathrm{d}x = \int_{x=0}^{1} -2\sqrt{1-x^2}\,\mathrm{d}x.$$

  This would be tricky as an integral, but we can spot that the integral of $\sqrt{1-x^2}$ as x goes from 0 to 1 is the area of a quarter of the unit circle, or $\frac{\pi}{4}$. Altogether, we get $-2(\frac{\pi}{4})$ or $-\frac{\pi}{2}$.

The total flux we get is the sum of the integrals of these three terms: $3 - \frac{\pi}{2}$.

1.2 A general approach

We can figure out what, in general, happens to the integral over a surface given by the implicit equation $f(x, y, z) = 0$. In that case, our first step would have been to rewrite this as $\frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z = 0$, letting us solve for $\mathrm{d}z$ as

$$\mathrm{d}z = -\frac{\partial f/\partial x}{\partial f/\partial z}\,\mathrm{d}x - \frac{\partial f/\partial y}{\partial f/\partial z}\,\mathrm{d}y.$$

Finally, a general 2-form like $M\,\mathrm{d}y \wedge \mathrm{d}z + N\,\mathrm{d}z \wedge \mathrm{d}x + P\,\mathrm{d}x \wedge \mathrm{d}y$ will simplify to

$$M\frac{\partial f/\partial x}{\partial f/\partial z}\,\mathrm{d}x \wedge \mathrm{d}y + N\frac{\partial f/\partial y}{\partial f/\partial z}\,\mathrm{d}x \wedge \mathrm{d}y + P\,\mathrm{d}x \wedge \mathrm{d}y = \left(\frac{M\frac{\partial f}{\partial x} + N\frac{\partial f}{\partial y} + P\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial z}}\right)\mathrm{d}x \wedge \mathrm{d}y.$$

If we have $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$, then the numerator here is $\mathbf{F} \cdot \boldsymbol{\nabla} f$, and the denominator can, for the sake of familiarity, be rewritten as $\boldsymbol{\nabla} f \cdot \mathbf{k}$.

All this is a reasonable approach, once again, *only* in the case where S has a one-to-one projection onto a region R in the xy-plane. But in that case, we deduce from this work that

$$\iint_S \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}A = \iint_R \frac{\mathbf{F} \cdot \boldsymbol{\nabla} f}{\boldsymbol{\nabla} f \cdot \mathbf{k}}\,\mathrm{d}x\,\mathrm{d}y.$$

There is another way to make this conclusion. We can combine two facts:

- The gradient vector ∇f is orthogonal to the surface $f(x, y, z) = 0$. So one choice of normal vector $\mathbf{n}$ is $\mathbf{n} = \frac{\nabla f}{\|\nabla f\|}$.
- We've already seen, when looking at scalar surface integrals, that under the same hypotheses we can replace $\mathrm{d}A$ by $\frac{\|\nabla f\|}{|\nabla f \cdot \mathbf{k}|}\,\mathrm{d}x\,\mathrm{d}y$.

We are both dividing and multiplying by $\|\nabla f\|$, and in the end, we get the same formula as above...

...except for one small difference. Why did $|\nabla f \cdot \mathbf{k}|$ in the scalar surface integral become $\nabla f \cdot \mathbf{k}$ in the vector surface integral?

This comes down to the orientation issue. First of all, $\nabla f \cdot \mathbf{k}$ cannot change sign, in the scenario where the projection onto the xy-plane is one-to-one: that would require $\nabla f \cdot \mathbf{k}$ to equal 0 at some point, which would mean the surface bending over itself. Therefore $\nabla f \cdot \mathbf{k}$ is either always positive or always negative. We can't say anything more about it, since $f(x, y, z) = 0$ and $-f(x, y, z) = 0$ are equally good descriptions of S, but have opposite signs of $\nabla f \cdot \mathbf{k}$.

However, it turns out (and this follows from our work with the differential forms, but can also be checked separately) that dropping the absolute value here is guaranteed to give the *upward* orientation for the flux integral: it corresponds to choosing $\mathbf{n}$ with a positive $\mathbf{k}$-component. So if we follow that convention, the absolute value is not necessary!

2 Other interesting examples

2.1 Boundary of a cube

Let $\mathbf{F} = x\,\mathbf{i} + z\,\mathbf{j} - y^2\,\mathbf{k}$. Let S be the *entire* boundary of the cube $[-1, 1] \times [-1, 1] \times [-1, 1]$. What is the outward flux of $\mathbf{F}$ across S?

Let's look at the top face of the cube, where $z = 1$ and $(x, y) \in [-1, 1] \times [-1, 1]$. Then:

- Reasoning from the
$$\iint \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}A$$
definition of the integral, the normal vector $\mathbf{n}$ is just $\mathbf{k}$, and integrating over the top boundary is just integrating over x and y each going from -1 to 1.

  Since $\mathbf{F} \cdot \mathbf{n} = \mathbf{F} \cdot \mathbf{k} = -y^2$, we get
$$\int_{x=-1}^{1} \int_{y=-1}^{1} -y^2\,\mathrm{d}y\,\mathrm{d}x.$$

- Reasoning from the
$$\iint M\,\mathrm{d}y \wedge \mathrm{d}z + N\,\mathrm{d}z \wedge \mathrm{d}x + P\,\mathrm{d}x \wedge \mathrm{d}y$$

definition of the integral, we can drop all the terms except $P\,\mathrm{d}x \wedge \mathrm{d}y = -y^2\,\mathrm{d}x \wedge \mathrm{d}y$, because $\mathrm{d}z = 0$. So we also get

$$\iint -y^2\,\mathrm{d}x \wedge \mathrm{d}y = \int_{x=-1}^{1}\int_{y=-1}^{1} -y^2\,\mathrm{d}y\,\mathrm{d}x.$$

In both cases, integrating $-y^2$ as y goes from -1 to 1 gives us $-\frac{1^3}{3} + \frac{(-1)^3}{3} = -\frac{2}{3}$.

What changes for the bottom face? Well,

- Reasoning from the first definition of the integral, we now want $\mathbf{n} = -\mathbf{k}$, so $\mathbf{F}\cdot\mathbf{n} = y^2$, instead. Everything else remains the same.
- Reasoning from the second definition of the integral, we still get a $-y^2\,\mathrm{d}x \wedge \mathrm{d}y$ term, but now we rewrite it as $y^2\,\mathrm{d}y \wedge \mathrm{d}x$ term before turning it into an unoriented integral. That's because $\mathrm{d}y \wedge \mathrm{d}x$ is the version of the 2-form whose orientation matches the orientation of the surface we want: the surface $z = -1$ with a *downward* normal vector.

Either way, since the sign flips, but the integral is otherwise the same, we will get $+\frac{2}{3}$, canceling out our previous $-\frac{2}{3}$.

What will happen on the other four faces? For the $y = 1$ and $y = -1$ faces of the cube, a similar cancellation is expected. On one of them, we take the normal vector $\mathbf{j}$, for $\mathbf{F}\cdot\mathbf{j} = z$. On the other one, we take the normal vector $-\mathbf{j}$, for $\mathbf{F}\cdot -\mathbf{j} = -z$.

For the $x = 1$ and $x = -1$ faces of the cube, we don't get cancellation, because the vector field itself has different behavior. When $x = 1$, the outward normal vector is $\mathbf{n} = \mathbf{i}$, and we get $\mathbf{F}\cdot\mathbf{n} = x = 1$. When $x = -1$, the outward normal vector is $\mathbf{n} = -\mathbf{i}$, and we get $\mathbf{F}\cdot\mathbf{n} = -x = 1$. So both faces have

$$\iint \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}A = \int_{y=-1}^{1}\int_{z=-1}^{1} \mathrm{d}z\,\mathrm{d}y = 4.$$

The total flux across the entire surface of the cube is $4 + 4 = 8$.

2.2 The Möbius strip, again

A few lectures ago, we discussed a parameterization of the Möbius strip like the one below:[2]

$$\mathbf{r}(u,v) = \left((5 + v\cos u)\cos 2u, (5 + v\cos u)\sin 2u, v\sin u\right), \qquad (u,v) \in [0,\pi]\times[-1,1].$$

What happens if we try to take the flux integral of $\mathbf{F} = \mathbf{k}$ across this vector field?

The first step of taking this integral is not very fun. We compute $\mathbf{k}\cdot\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right)$ by taking the determinant

$$\det\begin{bmatrix} 0 & 0 & 1 \\ (-v\sin u)\cos 2u - 2(5 + v\cos u)\sin 2u & (-v\sin u)\sin 2u + 2(5 + v\cos u)\cos 2u & v\cos u \\ \cos u\cos 2u & \cos u\sin 2u & \sin u \end{bmatrix}$$

[2] I have changed the parameterization slightly to make taking the derivatives, if you will believe it, slightly easier.

and the only good thing that happens is that the actual determinant step is easy because of the simple first row. What's more, there are two identical $(-v \sin u) \cos u \sin 2u \cos 2u$ terms that cancel, and we end up left with

$$-2(5 + v\cos u)\cos u \sin^2 2u - 2(5 + v\cos u)\cos u \cos^2 2u$$

whcih simplfiies to just $-2(5 + v\cos u)\cos u$ or $-10\cos u - 2v\cos^2 u$.

Now we must integrate:

$$\int_{u=0}^{\pi}\int_{v=-1}^{1} (-10\cos u - 2v\cos^2 u)\,\mathrm{d}v\,\mathrm{d}u = \int_{u=0}^{\pi} -20\cos u\,\mathrm{d}u = -20\sin u\Big|_{u=0}^{\pi} = 0.$$

Okay, good, so the flux integral of $\mathbf{k}$ across the Möbius strip is 0.[3] Now for the hard question: what does this mean?

To interpret a flux, we must first figure out which way the normal vector points. We don't really want to think about the full expression for $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ here, so let's do a test case: let's figure out the normal vector at the point $(5, 0, 0)$, which happens to be both $\mathbf{r}(0, 0)$ and $\mathbf{r}(\pi, 0)$.

Setting $v = 0$ in $\frac{\partial \mathbf{r}}{\partial u}$ simplifies it to $-10\sin 2u\,\mathbf{i} + 10\cos 2u\,\mathbf{j}$. When we follow up by setting $u = 0$, we get $\frac{\partial \mathbf{r}}{\partial u}\big|_{u=0,v=0} = 10\,\mathbf{j}$. In the case of $\frac{\partial \mathbf{r}}{\partial v}$, the variable v already does not show up; setting $u = 0$, we get $\frac{\partial \mathbf{r}}{\partial u}\big|_{u=0,v=0} = \mathbf{i}$. Therefore $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ is equal to $10\,\mathbf{j} \times \mathbf{i} = -10\,\mathbf{k}$ when $u = v = 0$. (The unit normal $\mathbf{n}$ is just $-\mathbf{k}$.)

Something concerning happens, though. What if we try $u = \pi$, instead? This is the same point $(5, 0, 0)$, but we will get $\frac{\partial \mathbf{r}}{\partial v}\big|_{u=0,v=0} = -\mathbf{i}$, instead, because $\cos\pi = -\cos 0$. The result is that $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = 10\,\mathbf{k}$, and the normal vector $\mathbf{n}$ points in the opposite direction!

The concerning thing that happens is that the Möbius strip is not orientable. If you imagine taking the normal vector at $u = 0$, and following it along as u increases, it will vary continuously, tilting as the Möbius strip twists until it's horizontal at $u = \frac{\pi}{2}$. Then, it will keep turning in the same direction until it ends up being the opposite of its previous orientation when $u = \pi$ and it comes back around.

This is just a very fancy calculus way of saying that there's no way to choose a "front side" and "back side" for the Möbius strip, because it only has one side!

In particular, it is very hard to make sense of our flux integral. The reason it is 0 is because for $0 \le u \le \pi/2$, our normal vector points downward and we get a negative flux; for $\pi/2 \le u \le \pi$, our normal vector points upward and we get a positive flux.

[3] This is not concerning in and of itself. The flux integral of $\mathbf{k}$ across many surfaces is 0; for example, across the unit sphere.