Math 3204: Calculus IV[1]	Mikhail Lavrov
Lecture 22: The curl of a vector field	
October 31, 2024	*Kennesaw State University*

1 Rotation vectors

So far, we have seen vectors used for two things: to indicate a **point** (in the plane or in space) or to indicate a **direction with magnitude**. It's useful to keep track of the difference: for example, if a curve C is parameterized by a function $\mathbf{r}(t)$, then a particular value like $\mathbf{r}(0)$ is a point; the derivative $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\Big|_{t=0}$ is a direction with magnitude. To keep track of the difference in these lecture notes, I have been using the notation (x, y, z) for points, and $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ for directions with magnitude.

A third way[2] to interpret a vector, this time **limited to vectors in** $\mathbb{R}^3$, is as a rotation vector.

To see why rotations in $\mathbb{R}^3$ have exactly the right number of degrees of freedom (unlike, say, rotations in $\mathbb{R}^2$), think about it this way:

- If you're spinning the plane $\mathbb{R}^2$ about the origin, all we need to say about the spin is "which way?" (clockwise or counterclockwise) and "how quickly?" This is described by a single real number (positive or negative).
- If you want to spin an object in $\mathbb{R}^3$ about the origin, you need to first pick an **axis of rotation**. Only then can you answer questions like "which way?" and "how quickly?"

A rotation vector does this as follows. Its direction tells us the axis of rotation, but *also* we adopt the convention that, from the point of view of an observer facing the vector head-on, the rotation is counterclockwise. Then, the magnitude of the rotation vector tells us how quickly we're spinning $\mathbb{R}^3$ about that axis—for example, in units of radians per second.

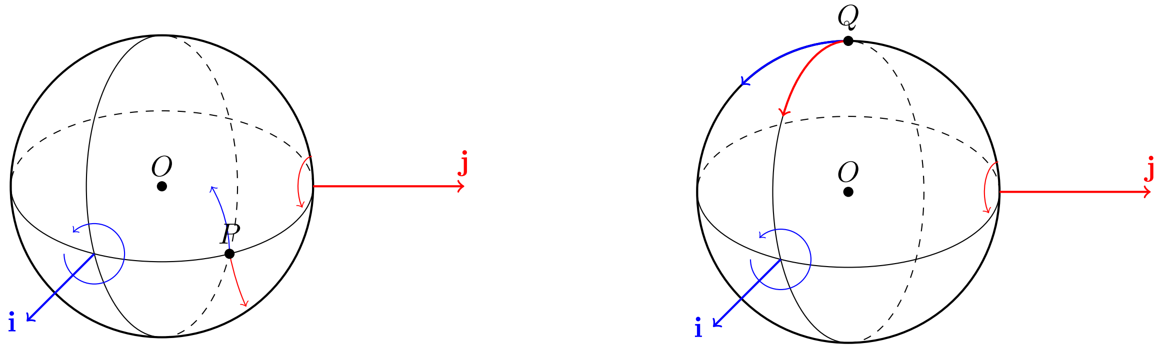
One of the bizarre things about this approach to rotation is that it plays nicely with vector addition! Suppose $\mathbf{a}$ and $\mathbf{b}$ are two rotation vectors. Then $\mathbf{a} + \mathbf{b}$ is the rotation ector that's the result of the rotations described by $\mathbf{a}$ and $\mathbf{b}$ happening *at the same time*.

Consider, for example, what happens when we take a unit sphere, and simultaneously apply the rotations described by rotation vector $\mathbf{i}$ and rotation vector $\mathbf{j}$? (See Figure 1.)

If P is the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$ at the surface of the sphere, then the immediate effect of the first rotation (described by $\mathbf{i}$) is to push P *upward*—that's where a rotation about the x-axis, counterclockwise when viewed from its positive end, will spin a point in the xy-plane. Similarly, the second rotation (described by $\mathbf{j}$) will push P *downward*. Because P is equidistant from the x-axis and y-axis, these effects exactly cancel out, and P stays put. This tells us the axis of the combined rotation: it is the line through the origin O and P. (See Figure 1a.)

[1] This document comes from an archive of the Math 3204 course webpage: http://misha.fish/archive/3204-fall-2024

[2] But not a third notation—partially because we've run out of ways to denote vectors, and partially because the same kinds of objects will situationally be used for different purposes.



(a) Determining the axis $\overrightarrow{OP}$ (b) Determining the rate of rotation

Figure 1: Combining the two rotations described by $\mathbf{i}$ (blue) and $\mathbf{j}$ (red)

Now look at what happens to the point $Q = (0, 0, 1)$. From the point of view of vector $\mathbf{i}$, the point Q lies on a circle of radius 1 in the yz-plane, centered at the origin. A counterclockwise rotation (viewed from the positive x-axis) will initially move Q in the $-\mathbf{j}$ direction. Meanwhile, from the point of view of $\mathbf{j}$, the point Q lies on a circle of radius 1 in the xz-plane, and its rotation will initially move Q in the $\mathbf{i}$ direction. Adding these together means that the initial movement of Q will be by the vector $\mathbf{i} - \mathbf{j}$: the combined rotation will be $\sqrt{2}$ times as fast as either of the individual rotations. It is also counterclockwise when viewed from P. (See Figure 1b.)

Putting these together, we see that the combined rotation is described by a vector parallel to $\overrightarrow{OP}$, in the same direction, with magnitude $\sqrt{2}$. This vector is exactly $\mathbf{i} + \mathbf{j}$, the sum of our two rotation vectors!

Now all you have to do is take my word for it that this happens in general.

2 The curl of a vector field

Imagine a tiny leaf tossed about by hurricane-force winds; $\mathbf{F}$ is the velocity field of the air. Then $\mathbf{F}$ itself is telling us where the leaf will be in the next instant—but that's not the whole story.[3] Tiny changes in $\mathbf{F}$ on a small scale (i.e. the derivative of $\mathbf{F}$) will tell us how the leaf *spins* when it occupies a point (x, y, z). The rotation vector describing this spin is the **curl** of $\mathbf{F}$ at (x, y, z).

We have already seen this for 2-dimensional vector fields: the "circulation density" or "curl" of $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ is given by the expression $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$. This expression can be lifted up to the third dimension to tell us the rotation vector in one specific case: when $\mathbf{F}$, despite being a 3-dimensional vector field,

1. has no $\mathbf{k}$-component (the velocity field is always parallel to the xy-plane), **and**
2. does not depend on z (the velocity field is the same at every height).

In such a scenario, the quantity $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ tells us the rate of counterclockwise rotation in the

[3] Actually, there's three parts to the story; the third part, which we will not talk about, is the *strain* on the leaf, which will tell us if it gets crushed or pulled apart by the wind.

xy-plane. In three dimensions, to represent such a rotation, we use the rotation vector $(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\,\mathbf{k}$.

In general, though, there will also be effects that spin the leaf about other axes. Because rotations add, we can just add up these effects. This leads us to the formula

$$\operatorname{curl}(\mathbf{F}) = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.$$

We have already seen this expression before, sort of: it comes from the component test for seeing if $\mathbf{F}$ is a gradient field. To remind you: in that case, there's a scalar function $f\colon \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F} = \boldsymbol{\nabla} f$. In other words, $M = \frac{\partial f}{\partial x}$, $N = \frac{\partial f}{\partial y}$, and $P = \frac{\partial f}{\partial z}$. In such a scenario, the components of $\operatorname{curl}(\mathbf{F})$ should be 0 because they're differences like $\frac{\partial}{\partial y}(\frac{\partial f}{\partial z}) - \frac{\partial}{\partial z}(\frac{\partial f}{\partial y})$.

3 The "del" operator

Here's another way to look at curl. Define the operator $\boldsymbol{\nabla}$ (or "del") to be

$$\boldsymbol{\nabla} = \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}.$$

This is a bit of a notational trick to help us describe several actual differential operators in consistent ways.

This $\boldsymbol{\nabla}$ is consistent with our notation $\boldsymbol{\nabla} f$ for the gradient vector of f:

$$\boldsymbol{\nabla} f = \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right) f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}.$$

The curl can now be written as $\boldsymbol{\nabla} \times \mathbf{F}$ instead of $\operatorname{curl}\mathbf{F}$, because

$$\boldsymbol{\nabla} \times \mathbf{F} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{bmatrix} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.$$

That is, by applying the rule for taking a cross product, we end up getting the definition of the curl! (For example, the first positive diagonal in this determinant is $\mathbf{i}\,\frac{\partial}{\partial y}\,P$, or $\frac{\partial P}{\partial y}\,\mathbf{i}$.)

The component test can now be summarized by the rule that for all functions $f\colon \mathbb{R}^3 \to \mathbb{R}$, we have

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = \mathbf{0}.$$

Let's look at an example, and use this formula to find the curl of $\mathbf{F} = xyz\,\mathbf{i} + xe^z\,\mathbf{j} + y\arcsin z\,\mathbf{k}$. Writing down

$$\boldsymbol{\nabla} \times \mathbf{F} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & xe^z & y\arcsin z \end{bmatrix},$$

we get:

- Three positive diagonal products. These are $\mathbf{i}\,\frac{\partial}{\partial y}\,y\arcsin z = \arcsin z\,\mathbf{i}$, then $\mathbf{j}\,\frac{\partial}{\partial z}\,xyz = xy\,\mathbf{j}$, then finally $\mathbf{k}\,\frac{\partial}{\partial x}\,xe^z = e^z\,\mathbf{k}$.
- Three negative diagonal products. These are $\mathbf{k}\,\frac{\partial}{\partial y}\,xyz = xz\,\mathbf{k}$, then $\mathbf{i}\,\frac{\partial}{\partial z}\,xe^z = xe^z\,\mathbf{i}$, then finally $\mathbf{j}\,\frac{\partial}{\partial x}\,y\arcsin z = 0$.

Adding these together, we get the final answer:

$$\nabla \times \mathbf{F} = (\arcsin z - xe^z)\,\mathbf{i} + xy\,\mathbf{j} + (e^z - x^z)\,\mathbf{k}.$$

Let me end with a few notes on terminology.

- ∇ "del" is also called "nabla" after a kind of triangular harp, which it looks like.
- $\operatorname{curl}\mathbf{F}$ is also sometimes called the "rotor of $\mathbf{F}$" (written $\operatorname{rot}\mathbf{F}$), short for the "rate of rotation" of $\mathbf{F}$.
- In the context of our wind example, where $\mathbf{F}$ is the velocity field of a fluid like air or water, $\nabla \times \mathbf{F}$ is called the "vorticity field" (from the word "vortex"). This is part of what makes $\mathbf{F}$ "turbulent".

  That's right: when you're on an airplane, and there's an announcement to stay in your seats because the flight will be experiencing some turbulence, what they're telling you is that the vorticity field of the air up ahead has unusually high magnitude.