Math 3204: Calculus IV[1]	Mikhail Lavrov
Lecture 23: Introduction to Stokes' theorem	
November 5, 2024	*Kennesaw State University*

1 From Green's theorem to Stokes' theorem

Here is a question you should be able to answer from what you know.

Let $\mathbf{F}$ be a three-dimensional vector field, and let S be a surface that happens to be entirely contained in the xy-plane, oriented with normal vectors facing upward. What is the easiest way to take the flux integral of $\mathbf{F}$ across S?

The answer is that in this case,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}A = \iint_S \mathbf{F}\cdot\mathbf{k}\,\mathrm{d}x\,\mathrm{d}y.$$

where the second integral is just a completely normal double integral of the scalar $\mathbf{F}\cdot\mathbf{k}$ (the $\mathbf{k}$-component of $\mathbf{F}$) with respect to x and y.

There's the intuitive reason for why this is true: $\mathbf{k}$ is the normal vector to S in the correct direction, and $\mathrm{d}S$ and $\mathrm{d}x\,\mathrm{d}y$ both represent area elements in the xy-plane.

We can also prove this formally using the techniques we know. Whenever S lies on the graph of an implicit equation $f(x,y,z)=0$ above a region R in the xy-plane, we can use this to convert the flux integral across S into a double integral over R. In this case, our implicit equation is "$z=0$", and the region R in the xy-plane is S itself. Normally, we'd be integrating $\mathbf{F}\cdot\frac{\nabla f}{\nabla f\cdot\mathbf{k}}$. In this case, $\nabla f = 0\,\mathbf{i}+0\,\mathbf{j}+1\,\mathbf{k}=\mathbf{k}$, so $\nabla f\cdot\mathbf{k}=1$, and the whole thing just simplifies to $\mathbf{F}\cdot\mathbf{k}$.

Now, a double integral of the $\mathbf{k}$-component of some vector field has already come up once for us—in the statement of Green's theorem for circulation integrals. In that setting, if S is a region in the xy-plane, $\mathbf{F}=M\,\mathbf{i}+N\,\mathbf{j}$ is a *two*-dimensional vector field, and C is the counterclockwise boundary of S, then we have

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_S \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y.$$

In the two-dimensional world, we thought of $\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}$ as the curl of $\mathbf{F}$. Now that we're thinking in three dimensions, it is merely the $\mathbf{k}$-component of curl: it is $(\nabla\times\mathbf{F})\cdot\mathbf{k}$.

Putting this together with our initial observation today, we conclude: when S is an upward-oriented surface in the xy-plane bounded by the counterclockwise curve C, we have

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{k}\,\mathrm{d}x\,\mathrm{d}y = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,\mathrm{d}A.$$

[1]This document comes from an archive of the Math 3204 course webpage: http://misha.fish/archive/3204-fall-2024

What we're seeing on the right-hand side is "the flux integral of the curl of $\mathbf{F}$ across S". That's a mouthful, so for simplicity, we will just call this integral the **curl integral** of $\mathbf{F}$ over S.

The new topic we are moving into next is Stokes' theorem. This theorem says that the equation we just wrote down for surfaces in the xy-plane holds more generally. If S is any surface in $\mathbb{R}^3$ and C is its *compatibly oriented* boundary, and if $\mathbf{F}$ is any vector field in $\mathbb{R}^3$, then the following two things are equal:

1. The circulation of $\mathbf{F}$ around C, $\displaystyle\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.
2. The curl integral of $\mathbf{F}$ over S, $\displaystyle\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}A$.

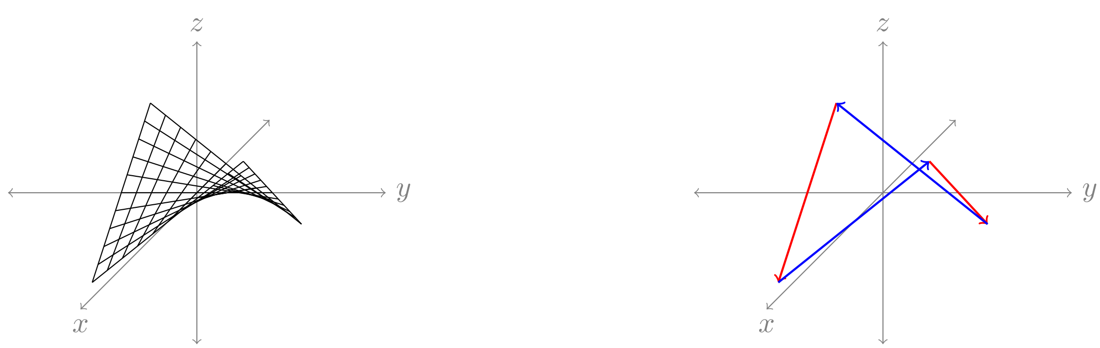
2 Finding boundaries

Before we apply Stokes' theorem in practice (something which we'll focus on in the next lecture) we should think about the objects we apply it to: surfaces and their boundaries.

This is where we are truly rewarded for sticking with *rectangular-domain* parameterizations for our surfaces. It turns out that when a surface S is given the counterclockwise parameterization $\mathbf{r}\colon [a, b] \times [c, d] \to \mathbb{R}^3$, then we get parameterizations of its boundary for free—by setting the parameters equal to their minimum and maximum values!

2.1 The hyperboloid

Let's start with a straightforward example where nothing weird happens: the hyperboloid parameterized by $\mathbf{r}(u, v) = (u, v, uv)$ with $(u, v) \in [-1, 1] \times [-1, 1]$. The surface itself is shown in Figure 1a.



(a) The surface S (b) The boundary of S

Figure 1: The hyperboloid and its boundary

The boundary of the rectangle $[a, b] \times [c, d]$ in the uv-plane is given by four line segments: from (a, c) to (b, c) to (b, d) to (a, d) to (a, c). To obtain the boundary of our surface, we will take the image of those line segments! That is done by fixing one of the variables u or v to a constant, and letting the other vary with t:

- The first segment of the boundary, where (u, v) goes from (a, c) to (b, c), is given by $\mathbf{r}_1(t) = \mathbf{r}(t, c)$, where $t \in [a, b]$.

  In our example, we get $\mathbf{r}_1(t) = \mathbf{r}(t, -1) = (t, -1, -t)$ where $t \in [-1, 1]$. This is a line segment from $(-1, -1, 1)$ to $(1, -1, -1)$.

- The second segment of the boundary, where (u, v) goes from (b, c) to (b, d), is given by $\mathbf{r}_2(t) = \mathbf{r}(b, t)$, where $t \in [c, d]$.

  In our example, we get $\mathbf{r}_2(t) = \mathbf{r}(1, t) = (1, t, t)$ where $t \in [-1, 1]$. This is a line segment from $(1, -1, -1)$ to $(1, 1, 1)$.

- For the third segment of the boundary, we have to get a bit tricky, because we want (u, v) to go from (b, d) to (a, d). That means that the u-coordinate has to decrease from b to a. The simplest way to write this is to have $\mathbf{r}_3(t) = \mathbf{r}(-t, d)$, where $t \in [-b, -a]$.

  In our example, we get $\mathbf{r}_3(t) = \mathbf{r}(-t, 1) = (-t, 1, -t)$, where $t \in [-1, 1]$. This is a line segment from $(1, 1, 1)$ to $(-1, 1, -1)$.

- Finally, for the fourth segment of the boundary, we have to do the same thing again, with the other variable. We take $\mathbf{r}_4(t) = \mathbf{r}(a, -t)$, where $t \in [-d, -c]$.

  In our example, we get $\mathbf{r}_4(t) = \mathbf{r}(-1, -t) = (-1, -t, t)$, where $t \in [-1, 1]$. This is a line segment from $(-1, 1, -1)$ to $(-1, -1, 1)$, the point where we started.

The four segments we get are shown in Figure 1b. There's one thing to watch out for: when going around the boundary of $[a, b] \times [c, d]$, we want to change u first, then v. Doing it in the other order would cause problems with orientations later.

2.2 The cylinder

There are a few important things to watch out for. The first is that sometimes two of these boundaries cancel out.

For example, suppose we take the cylinder of height 3 parameterized by $\mathbf{r}(u, v) = (\cos u, \sin u, v)$, where $(u, v) \in [0, 2\pi] \times [0, 3]$.

- We begin with $\mathbf{r}_1(t) = \mathbf{r}(t, 0) = (\cos t, \sin t, 0)$, where $t \in [0, 2\pi]$: a parameterization of the unit circle in the xy-plane.
- Next, we have $\mathbf{r}_2(t) = \mathbf{r}(2\pi, t) = (\cos 2\pi, \sin 2\pi, t) = (1, 0, t)$, where $t \in [0, 3]$: a line segment from $(1, 0, 0)$ to $(1, 0, 3)$.
- Next, we have $\mathbf{r}_3(t) = \mathbf{r}(-t, 3) = (\cos -t, \sin -t, 3) = (\cos t, -\sin t, 3)$, where $t \in [-2\pi, 0]$: another circle, this time in the plane $z = 3$.
- Finally, we have $\mathbf{r}_4(t) = \mathbf{r}(0, -t) = (\cos 0, \sin 0, -t) = (1, 0, -t)$, where $t \in [-3, 0]$: a line segment from $(1, 0, 3)$ to $(1, 0, 0)$.

Parameterizations $\mathbf{r}_1(t)$ and $\mathbf{r}_3(t)$ are what we expect to see: they are the curves along the top and bottom of the cylinder. In fact, they're *all* we expected. What's up with $\mathbf{r}_2(t)$ and $\mathbf{r}_4(t)$?

These two parameterizations describe the same line segment, but with opposite orientation. If we were to take any kind of line integral along the boundary, the effects from $\mathbf{r}_2(t)$ and $\mathbf{r}_4(t)$ would cancel out. Once we realize this, we can drop them entirely—they do not contribute anything to the boundary!

Here, because we know what cylinders look like, we know what to expect. In general, we should be on the lookout for parts of the boundary where this effect happens.

2.3 The cone

For a third example, let's take a cone of radius 1 and height 3, oriented so that its base is on the xy-plane (centered at the origin) and its tip is at the point $(0, 0, 3)$. This cone can be parameterized by

$$\mathbf{r}(u, v) = (u \cos v, u \sin v, 3 - 3u), \qquad (u, v) \in [0, 1] \times [0, 2\pi].$$

What happens when we try to find the boundary?

- We begin with $\mathbf{r}_1(t) = \mathbf{r}(t, 0) = (t \cos 0, t \sin 0, 3 - 3t)$, where $t \in [0, 1]$. This simplifies to $\mathbf{r}_1(t) = (t, 0, 3 - 3t)$, and is a line segment from $(0, 0, 3)$ to $(1, 0, 0)$.
- Next, $\mathbf{r}_2(t) = \mathbf{r}(1, t) = (\cos t, \sin t, 0)$, where $t \in [0, 2\pi]$. This is a parameterization of the unit circle: the base of the cone.
- Next, $\mathbf{r}_3(t) = \mathbf{r}(-t, 2\pi) = (-t \cos 2\pi, -t \sin 2\pi, 3 - 3t)$, where $t \in [-1, 0]$. This simplifies to $\mathbf{r}_3(t) = (-t, 0, 3 - 3t)$, where $t \in [-1, 0]$, and is a line segment from $(1, 0, 0)$ to $(0, 0, 3)$, canceling out with $\mathbf{r}_1(t)$.
- Finally, $\mathbf{r}_4(t) = \mathbf{r}(0, -t) = (0 \cos(-t), 0 \sin(-t), 3 - 3(0))$, where $t \in [-2\pi, 0]$. This simplifies to $\mathbf{r}_4(t) = (0, 0, 3)$: a constant!

What happened here is that $\mathbf{r}_4(t)$ shows one of the ways a parameterization isn't required to be injective: an entire segment on the boundary of the domain $[a, b] \times [c, d]$ can be sent to a single point. Here, that point is the tip of the cone.

In such a case, any line integral of the portion of the boundary parameterized by $\mathbf{r}_4(t)$ will simplify to 0 all by itself. (One reason for that: since $\mathbf{r}_4(t)$ is a constant, we get $\frac{d\mathbf{r}_4}{dt} = \mathbf{0}$, which makes the integral 0 as well.)

We can ignore $\mathbf{r}_4(t)$ for this reason. We can also ignore $\mathbf{r}_1(t)$ and $\mathbf{r}_3(t)$, because they cancel. The only boundary of the cone worth thinking about is $\mathbf{r}_2(t) = (\cos t, \sin t, 0)$, where $t \in [0, 2\pi]$.

3 Compatible orientations

The circulation integral and the curl integral in Stokes' theorem are both oriented integrals: their sign depends on the orientation of C and of S. This means that the two integrals will only be equal if the orientations of C and S are chosen to be compatible. But what makes the orientations compatible?

The first piece of good news: if we use the approach from the previous section to go from a parameterization of S to a parameterization of C, then the orientations we get will automatically be compatible with each other. (For this to happen, it's important that $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4$ are defined correctly: in the uv-plane, we want to take the boundary going counterclockwise, from (a, c) to (b, c) to (b, d) to (a, d) to (a, c), instead of the reverse order.) This means that in many situations, we will not have to worry about incompatibility between our orientations.

The underlying geometry, however, is worth understanding.

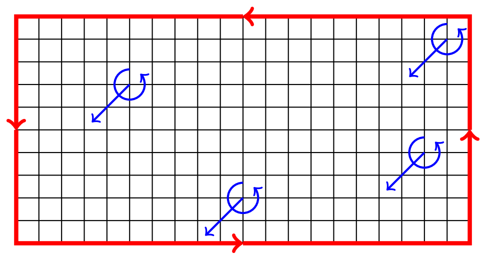


Figure 2: Compatible orientations of a rectangle and its boundary. You should interpret this 2D image as depicting a 3D situation with the blue vectors pointing toward you.

For the most part so far, we've thought of orienting a surface as picking a normal vector for that surface—but a normal vector at a point corresponds to a direction of rotation at that point. We follow the same convention that we've used in every such instance so far: a normal vector $\mathbf{n}$ corresponds to the direction of rotation that appears counterclockwise to a viewer facing the normal vector head-on.

As we move around the surface, the normal vector changes continuously. When we bring the normal vector close to the boundary of a surface, the direction of rotation indicates a direction of following that boundary. We can see this in Figure 2: when we draw the blue normal vector in close proximity to the red boundary, the blue arrows and the red arrows should be pointing in similar directions.

This is compatibility of orientation. It's a compatibility of orientation we didn't really have to worry about with Green's theorem, in most cases. The normal vector $\mathbf{k}$ points upward from the xy-plane, and this is compatible with a boundary that goes around the region counterclockwise. Dealing with regions that have holes is a bit more difficult, but all that happens in the plane is that we subtract the counterclockwise integral around each hole—or add the clockwise integral.

In three dimensions, we have to be more careful, because there's no objective notion of "clockwise" or "counterclockwise".

For example, let's return to the cylinder from earlier in this lecture. If the surface S is parameterized by $\mathbf{r}(u, v) = (\cos u, \sin u, v)$ where $(u, v) \in [0, 2\pi] \times [0, 3]$, then

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{bmatrix} = \cos u \, \mathbf{i} + \sin u \, \mathbf{j}$$

which is a normal vector pointing outward from the cylinder. The calculations we did earlier gave us a boundary consisting of two parts: a circle C_1 parameterized by $\mathbf{r}_1(t) = (\cos t, \sin t, 0)$, where $t \in [0, 2\pi]$, and a circle C_3 parameterized by $\mathbf{r}_3(t) = (\cos t, -\sin t, 3)$, where $t \in [-2\pi, 0]$. (Actually, we can clean C_3 up a little and change the range of t to $[0, 2\pi]$, since sin and cos both have a period of 2π.)

If we view both C_1 and C_3 from above, it will look like we're going counterclockwise around C_1, but clockwise around C_3. This seems strange! It might seem less strange to you if you imagine taking the rectangle in Figure 2, and wrapping it around a cylinder, seeing what happens to the orientations.