Math 3204: Calculus IV[1] — Mikhail Lavrov

# Lecture 4: Substitution with three variables

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## 1 Wedge products

Last time, we used

$$\iint_R f(x,y)\,\mathrm{d}x \wedge \mathrm{d}y$$

as simply notation to show that our integral is an oriented integral over the oriented region $R$. This time, we'll see that $\wedge$ is also an operation on differential elements that gives us another way to perform substitutions. The symbol $\wedge$ is called a "wedge product", and this is a helpful name: it's a *product*, and just like all products, it distributes over addition.

It also follows several other rules:

1. For a function $f$ of $x$ and $y$, we have $\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y$. This is a version of the chain rule. More commonly, we will use it to write, say, $\mathrm{d}x = \frac{\partial x}{\partial u}\,\mathrm{d}u + \frac{\partial x}{\partial v}\,\mathrm{d}v$.
2. $\mathrm{d}v \wedge \mathrm{d}u = -(\mathrm{d}u \wedge \mathrm{d}v)$. This is because our wedge products keep track of orientation, and the $vu$-plane has the opposite orientation from the $uv$-plane.
3. As a special case, $\mathrm{d}u \wedge \mathrm{d}u = 0$. (This is the only way that the second rule could be true, because it tells us that $\mathrm{d}u \wedge \mathrm{d}u = -(\mathrm{d}u \wedge \mathrm{d}u)$ when we set $u = v$.)

The point of these rules is that to integrate with respect to $u$ and $v$ rather than $x$ and $y$, all we have to do is write $\mathrm{d}x \wedge \mathrm{d}y$ in terms of $\mathrm{d}u \wedge \mathrm{d}v$.

Let's see this on two examples. First, our transformation of the ellipse from the previous lecture, with $x = u + \frac{1}{2}v$ and $y = \frac{1}{2}v$. Here,

$$\mathrm{d}x \wedge \mathrm{d}y = (\mathrm{d}u + \tfrac{1}{2}\,\mathrm{d}v) \wedge (\tfrac{1}{2}\,\mathrm{d}v) = \tfrac{1}{2}\,\mathrm{d}u \wedge \mathrm{d}v + \tfrac{1}{4}\,\mathrm{d}v \wedge \mathrm{d}v = \tfrac{1}{2}\,\mathrm{d}u \wedge \mathrm{d}v.$$

Next, let's look at a more complicated example: $x = u^3 v$ and $y = u + v$. Here, $\mathrm{d}x = 3u^2 v\,\mathrm{d}u + u^3\,\mathrm{d}v$ and $\mathrm{d}y = \mathrm{d}u + \mathrm{d}v$, so $\mathrm{d}x \wedge \mathrm{d}y$ becomes

$$(3u^2 v\,\mathrm{d}u + u^3\,\mathrm{d}v) \wedge (\mathrm{d}u + \mathrm{d}v).$$

When we distribute, the $3u^2 v\,\mathrm{d}u \wedge \mathrm{d}u$ and $u^3\,\mathrm{d}v \wedge \mathrm{d}v$ terms become 0, and we're left with only the "cross" terms:

$$3u^2 v\,\mathrm{d}u \wedge \mathrm{d}v + u^3\,\mathrm{d}v \wedge \mathrm{d}u = (3u^2 v - u^3)\,\mathrm{d}u \wedge \mathrm{d}v.$$

You should feel free to use either method—the Jacobian determinant, or the wedge product—to perform substitutions. They will give the same answer, as long as you remember that for a non-oriented integral, you need to take the absolute value of the scaling factor on the differential!

[1]This document comes from an archive of the Math 3204 course webpage: http://misha.fish/archive/3204-fall-2024

## 2 The 3-variable Jacobian

In earlier courses, you learned about $u$-substitutions, which replace $x$ with another variable $u$. In the previous lecture, you learned about $uv$-substitutions, which replace two variables $(x, y)$ with variables $(u, v)$. Finally, today we will learn about $uvw$-substitutions, which replace three variables $(x, y, z)$ with variables $(u, v, w)$.

For the same reasons that it was true in two dimensions, every $uvw$-substitution also comes with a scaling factor we will write as $\frac{\partial(x,y,z)}{\partial(u,v,w)}$, usually with an absolute value on it. But what is that factor?

One way to arrive at it is to use the same algebra of wedge products that we used in two variables. We begin by using the chain rule to write $\mathrm{d}x$ as $\frac{\partial x}{\partial u}\,\mathrm{d}u + \frac{\partial x}{\partial v}\,\mathrm{d}v + \frac{\partial x}{\partial w}\,\mathrm{d}w$, and the same for $y$ and $z$. Then, we simplify $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ by three rules:

- We distribute $\wedge$ over addition, and factor out identical differential terms.
- We can swap two differentials if we flip the sign, such as $\mathrm{d}v \wedge \mathrm{d}u = -(\mathrm{d}u \wedge \mathrm{d}v)$.
- A wedge product like $\mathrm{d}u \wedge \mathrm{d}u$, which multiplies a differential with itself, is 0.

If we simplify $(a\,\mathrm{d}u + b\,\mathrm{d}v) \wedge (c\,\mathrm{d}u + d\,\mathrm{d}v)$, we get $(ad - bc)\,\mathrm{d}u \wedge \mathrm{d}v$, which motivates the $2 \times 2$ determinant. If we simplify

$$(a\,\mathrm{d}u + b\,\mathrm{d}v + c\,\mathrm{d}w) \wedge (d\,\mathrm{d}u + e\,\mathrm{d}v + f\,\mathrm{d}w) \wedge (g\,\mathrm{d}u + h\,\mathrm{d}v + i\,\mathrm{d}w)$$

then we first get

$$\big((ae - bd)\,\mathrm{d}u \wedge \mathrm{d}v + (bf - ce)\,\mathrm{d}v \wedge \mathrm{d}w + (cd - af)\,\mathrm{d}w \wedge \mathrm{d}u\big) \wedge (g\,\mathrm{d}u + h\,\mathrm{d}v + i\,\mathrm{d}w)$$

and then finally $(aei + bfg + cdh - afh - bdi - ceg)\,\mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}w$. The factor in front is exactly the determinant of a $3 \times 3$ matrix:

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - afh - bdi - ceg.$$

Instead of arbitrary numbers $a, b, c, d, e, f, g, h, i$, our matrix will have the partial derivatives $\frac{\partial x}{\partial u}$ through $\frac{\partial z}{\partial w}$ in it. We define the 3-variable Jacobian determinant $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ to be

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}.$$

The non-oriented rule for substitution says the following. Let $\mathbf{f}\colon \mathbb{R}^3 \to \mathbb{R}^3$, with $\mathbf{f}(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$, be a bijection from a region $S$ in $uvw$-space to a region $R$ in $xyz$-space. Then for a function $g\colon \mathbb{R}^3 \to \mathbb{R}$,

$$\iiint_R g(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_S g(\mathbf{f}(u, v, w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w.$$

Again, there's an absolute value. Again, to drop the absolute value, we need to pass to oriented regions. Again, we mostly don't need to worry about oriented regions for now—and in fact, we will not need to have much intuition for oriented 3-dimensional regions at all, this semester. (Our notion of orientation for 2-dimensional regions will give us a notion of orientation for 2-dimensional surfaces in $\mathbb{R}^3$. Similarly, orientations of 3-dimensional regions would be useful if we were later going to think about 3-dimensional hypersurfaces in $\mathbb{R}^4$, which we won't.)

However, you might be curious: what is the analogue of "clockwise" and "counterclockwise" for 3-dimensional orientations? It is the "right-hand rule" you may have learned for cross products. Standard rectangular coordinates on $\mathbb{R}^3$ follow the right-hand rule, because the cross product $\mathbf{i} \times \mathbf{j}$ of the basis vectors $\mathbf{i} = (1, 0, 0)$ and $\mathbf{j} = (0, 1, 0)$ is the basis vector $\mathbf{k} = (0, 0, 1)$. A mirror image of $\mathbb{R}^3$ would follow a "left-hand rule" where $\mathbf{j} \times \mathbf{i} = \mathbf{k}$; it would have the opposite orientation.

## 3 Computing $3 \times 3$ determinants

There is a mnemonic called the **rule of Sarrus** for the determinant of a $3 \times 3$ matrix. First, extend the matrix to a $3 \times 5$ grid by repeating the first and second columns. Then, add the products of the three top-left-to-bottom-right diagonals (Figure 1, in red) and subtract the products of the three bottom-left-to-top-right diagonals (Figure 1, in blue).

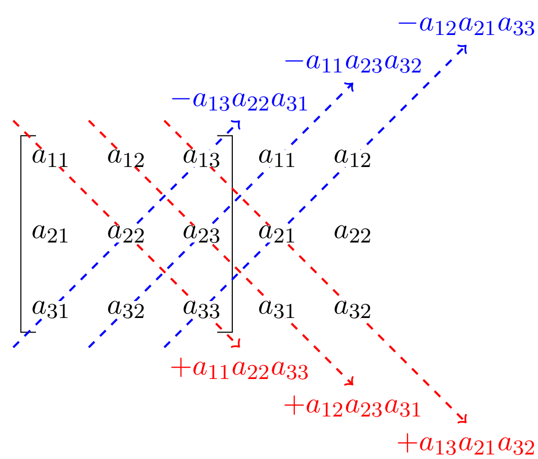


Figure 1: An illustration of the rule of Sarrus

This rule is a bad rule to learn if you're doing linear algebra, because it doesn't generalize to $n \times n$ determinants. However, we will not go higher than $3 \times 3$ in this class, so it's perfect for us.

There are also two special cases that we'll commonly encounter. First: if a matrix is upper triangular or lower triangular, its determinant is just the product of the entries on the main diagonal:

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} = \det \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33}.$$

A special case of this is a diagonal matrix, where only $a_{11}, a_{22}, a_{33}$ are nonzero. This occurs when our $uvw$-substitution replaces $x$ by a function only of $u$, $y$ by a function only of $v$, and $z$ by a function only of $w$.

Second, we often encounter $uvw$-substitutions which split up into two parts: say, $x, y$ are replaced by functions of $u, v$, and $z$ is replaced by a function of $w$. This gives us a Jacobian matrix with a block structure: we have a $2 \times 2$ matrix and a $1 \times 1$ matrix "joined diagonally", with 0's connecting them. In this case, the determinant is the product of the $2 \times 2$ determinant and the $1 \times 1$ entry remaining:

$$\det \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} = (a_{11}a_{22} - a_{12}a_{21})a_{33}.$$

## 4 A typical example

Here is a triple integral we might want to do by substitution:

$$\int_{x=-1}^{1} \int_{y=4-x}^{5-x} \int_{z=2y-1}^{2y+1} (x^2 + xy)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x.$$

How should we substitute? Well, $x$ seems totally fine as a variable, so let's keep $u = x$. The bounds on $y$ are $4 - x \le y \le 5 - x$, or $4 \le x + y \le 5$, so we might want to set $v = x + y$ to get $4 \le v \le 5$. Finally, the bounds on $z$ are $2y - 1 \le z \le 2y + 1$, or $-1 \le z - 2y \le 1$, so we might want to set $w = z - 2y$ to get $-1 \le w \le 1$.

To find the Jacobian determinant, we first want to solve for $x, y, z$ in terms of $u, v, w$. From the first equation, $x = u$. From the second equation, $y = v - x = v - u$. Finally, from the third equation, $z = w + 2y = w + 2(v - u)$. We have

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} = 1.$$

(This determinant is particularly easy to evaluate, because the matrix is lower-triangular!)

Since the Jacobian determinant is 1, we do not need to add any extra factors when substituting. Our last step is to rewrite $x^2 + xy$ as $u^2 + u(v - u) = uv$; then we can express our integral as

$$\int_{u=-1}^{1} \int_{v=4}^{5} \int_{w=-1}^{1} uv\, \mathrm{d}w\, \mathrm{d}v\, \mathrm{d}u.$$

While we're at it, here is another trick to keep in mind, because it will make many of our integrals easier. The integrand $uv$ is an odd function of $u$: when you replace $u$ by $-u$, it switches sign. (It's also an odd function of $v$, though that's not relevant here.) Also, our region, which is a cuboid $[-1, 1] \times [4, 5] \times [-1, 1]$, is symmetric about the $vw$-plane: it is unchanged when you replace $u$ by $-u$.

When we integrate an odd function of $u$ over such a region, the negative values of $u$ will exactly cancel out with the positive values of $u$, and we'll get 0. Therefore, even though we could do this integral the long way, we don't have to; we know that we'll get 0 at the end.

## 5 Cylindrical and spherical coordinates

With the power of the three-variable Jacobian, we can re-derive our rules for integration in cylindrical and spherical coordinates from first principles.

Consider this section **optional** when reading the notes: it gives some useful examples, but we've already derived the formulas in this section in a different way, and we won't have time to cover all of these examples in class.

### 5.1 Cylindrical coordinates

The substitution for cylindrical coordinates is $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. Here, our variables $r, \theta, z$ play the role of $u, v, w$ in a $uvw$-substitution: after all, $u$, $v$, and $w$ are just names!

The Jacobian determinant of this substitution is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix} = \det\begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This is one of those cases where our matrix has a $2\times 2$ block and a $1\times 1$ block. So we take the $2\times 2$ determinant: $\cos\theta(r\cos\theta) - (-r\sin\theta)(\sin\theta) = r\cos^2\theta + r\sin^2\theta = r$. Then, we multiply by the $1\times 1$ entry, which is just 1.

We've gotten $\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$, which means that technically, we should be replacing $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ by $|r|\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$. However, as a rule, we only work with nonnegative values of $r$, so $|r|$ and $r$ are always equal.

What if we use the wedge product to derive this substitution? Well, we have

$$\begin{aligned} \mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z &= (\cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta)\wedge(\sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta)\wedge\mathrm{d}z \\ &= (r\cos^2\theta\,\mathrm{d}r\wedge\mathrm{d}\theta - r\sin^2\theta\,\mathrm{d}\theta\wedge\mathrm{d}r)\wedge\mathrm{d}z \\ &= (r\cos^2\theta + r\sin^2\theta)\,\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}z \\ &= r\,\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}z. \end{aligned}$$

### 5.2 Spherical coordinates

Spherical coordinates are a more painful endeavor. Here, we have $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$. Taking partial derivatives gives us

$$\begin{aligned} \mathrm{d}x &= \sin\phi\cos\theta\,\mathrm{d}\rho - \rho\sin\phi\sin\theta\,\mathrm{d}\theta + \rho\cos\phi\cos\theta\,\mathrm{d}\phi \\ \mathrm{d}y &= \sin\phi\sin\theta\,\mathrm{d}\rho + \rho\sin\phi\cos\theta\,\mathrm{d}\theta + \rho\cos\phi\sin\theta\,\mathrm{d}\phi \\ \mathrm{d}z &= \cos\phi\,\mathrm{d}\rho - \rho\sin\phi\,\mathrm{d}\phi. \end{aligned}$$

From here, we can continue with the determinant approach:

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \det\begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix}.$$

This looks bad, but the rule of Sarrus only gives us two positive diagonals and two negative diagonals, because of the 0 entry. It turns out that they're easier to do in pairs:

- Adding the positive diagonal $(\sin\phi\cos\theta)(\rho\sin\phi\cos\theta)(-\rho\sin\phi)$ and subtracting the negative diagonal $(\rho\cos\phi\cos\theta)(\rho\sin\phi\cos\theta)(\cos\phi)$ simplifies to $-\rho^2\sin\phi\cos^2\theta(\sin^2\phi+\cos^2\phi)$ or just $-\rho^2\sin\phi\cos^2\theta$.
- Adding the positive diagonal $(-\rho\sin\phi\sin\theta)(\rho\cos\phi\sin\theta)(\cos\phi)$ and subtracting the negative diagonal $(-\rho\sin\phi\sin\theta)(\sin\phi\sin\theta)(-\rho\sin\phi)$ simplifies to $-\rho^2\sin\phi\sin^2\theta(\cos^2\phi+\sin^2\phi)$ or just $-\rho^2\sin\phi\sin^2\theta$.

Putting these two terms together lets us simplify further, to $-\rho^2\sin\phi(\cos^2\theta+\sin^2\theta)$ or just $-\rho^2\sin\phi$.

We have a negative sign in front; this is due to the order we chose for our variables. When we take the absolute value of the Jacobian, the negative sign goes away, but $\rho^2$ and $\sin\phi$ are both guaranteed to be positive. Therefore we always replace $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ by $\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$.

Doing the same thing again with wedge products would be needlessly extending our suffering, but let's combine the wedge product approach with another trick: start halfway, from our cylindrical coordinates. We already know that $\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z = r\,\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}z$. Well, in spherical coordinates, we have $z=\rho\cos\phi$ and $r=\rho\sin\phi$. Therefore $\mathrm{d}z=\cos\phi\,\mathrm{d}\rho-\rho\sin\phi\,\mathrm{d}\phi$, and $\mathrm{d}r=\sin\phi\,\mathrm{d}\rho+\rho\cos\phi\,\mathrm{d}\phi$. Continuing where we left off, we have

$$\begin{aligned}
\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z &= r\,\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}z\\
&= \rho\sin\phi(\sin\phi\,\mathrm{d}\rho+\rho\cos\phi\,\mathrm{d}\phi)\wedge\mathrm{d}\theta\wedge(\cos\phi\,\mathrm{d}\rho-\rho\sin\phi\,\mathrm{d}\phi)\\
&= -\rho\sin\phi(\sin\phi\,\mathrm{d}\rho+\rho\cos\phi\,\mathrm{d}\phi)\wedge(\cos\phi\,\mathrm{d}\rho-\rho\sin\phi\,\mathrm{d}\phi)\wedge\mathrm{d}\theta\\
&= -\rho\sin\phi(-\rho\sin^2\phi\,\mathrm{d}\rho\wedge\mathrm{d}\phi+\rho\cos^2\phi\,\mathrm{d}\phi\wedge\mathrm{d}\rho)\wedge\mathrm{d}\theta\\
&= -\rho\sin\phi(-\rho\sin^2\phi-\rho\cos^2\phi)\,\mathrm{d}\rho\wedge\mathrm{d}\phi\wedge\mathrm{d}\theta\\
&= \rho^2\sin\phi\,\mathrm{d}\rho\wedge\mathrm{d}\phi\wedge\mathrm{d}\theta.
\end{aligned}$$