Math 3204: Calculus IV[1] Mikhail Lavrov

Lecture 8: Vector line integrals

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1 The work integral

The vector line integral is the integral of a vector field $\mathbf{F}$ along an *oriented* curve C; it is denoted

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

If $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$, this integral is also written

$$\int_C M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}z.$$

We'll talk about why that is later.

We will begin from an application: the vector line integral of $\mathbf{F}$ along C is equal to the work done by a force $\mathbf{F}$ on a particle moving along C. We will derive what the vector line integral "should be" for this to be the true, and we will make that our general definition of the vector line integral.

1.1 Deriving the formula

We know that when $\mathbf{F}$ is a constant force, and an object moves along vector $\mathbf{s}$, then the work done on the object by $\mathbf{F}$ is given by the dot product $\mathbf{F} \cdot \mathbf{s}$.

In general, the path of the object may be more complicated, and the force isn't constant either. To understand the complicated case from the simple one, we'll take an approximation: we divide the curve C into many tiny segments, and compute the work done as the object travels along each segment. If a segment is very short, then the path of the object along it is approximately straight, and the force can't possibly change too much, either.

Say that the curve C is parameterized by $\mathbf{r}(t)$ where $t \in [a, b]$. Here, our curve is oriented, and it's important for $\mathbf{r}(t)$ to respect that: as t increases from a to b, $\mathbf{r}(t)$ must move from the start of the curve to the end, matching the orientation of the curve.

We divide the interval $[a, b]$ into many tiny intervals:

$$[a, b] = [t_0, t_1] \cup [t_1, t_2] \cup \cdots \cup [t_{n-1}, t_n]$$

where, for concreteness, $t_i = a + i\Delta t$ and $\Delta t = \frac{b-a}{n}$. At time t_i, the object is at position $\mathbf{r}(t_i)$, and the force acting on it is $\mathbf{F}(\mathbf{r}(t_i))$. Over the time interval $[t_i, t_{i+1}]$, the object *approximately* moves in a straight line along the vector $\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)$, and the force *approximately* stays equal to $\mathbf{F}(\mathbf{r}(t_i))$,

[1]This document comes from an archive of the Math 3204 course webpage: http://misha.fish/archive/3204-fall-2024

so the work done in that interval of time is *approximately* $\mathbf{F}(\mathbf{r}(t_i)) \cdot [\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)]$. The overall work done is

$$\sum_{i=0}^{n-1} \mathbf{F}(\mathbf{r}(t_i)) \cdot [\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)].$$

To make this sum more similar to a Riemann sum, write $\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)$ as $\frac{\mathbf{r}(t_{i+1})-\mathbf{r}(t_i)}{\Delta t} \cdot \Delta t$. Yet a third small-$\Delta t$ approximation: $\frac{\mathbf{r}(t_{i+1})-\mathbf{r}(t_i)}{\Delta t}$ is approximately equal to $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t_i)$. So the work done is approximately

$$\sum_{i=0}^{n-1} \mathbf{F}(\mathbf{r}(t_i)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t_i)\Delta t.$$

In the limit as $n \to \infty$, $\Delta t \to 0$, and the approximations should resolve to the true amount of work done. But the limit of this Riemann sum as $n \to \infty$ is an integral:

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \mathbf{F}(\mathbf{r}(t_i)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t_i)\Delta t = \int_{t=a}^{b} \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t.$$

Now, in order for the vector line integral to compute this important quantity from physics, we *define* the vector line integral as

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} := \int_{t=a}^{b} \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t.$$

Think of this notation for the vector line integral as a mnemonic for the operations we do to compute it; replacing $\mathrm{d}\mathbf{r}$ by $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t$ looks like the rule for integration by substitution. (It's not *really* a substitution, because we don't otherwise know how to make sense of an integral with $\mathrm{d}\mathbf{r}$.)

Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$, and let $\mathbf{r}(t) = (x(t), y(t), z(t))$. Then we can write this integral as

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{t=a}^{b} \left(M(\mathbf{r}(t))\frac{\mathrm{d}x}{\mathrm{d}t} + N(\mathbf{r}(t))\frac{\mathrm{d}y}{\mathrm{d}t} + P(\mathbf{r}(t))\frac{\mathrm{d}z}{\mathrm{d}t} \right) \mathrm{d}t.$$

For this reason, the vector line integral is sometimes written as

$$\int_C M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}z.$$

We will look at the quantity $M\,\mathrm{d}x + N\,\mathrm{d}y + P\,\mathrm{d}z$ a bit more closely later; it is called a **differential form** (more precisely, a 1-form, because there's one differential on every term). The idea is that once you do a change of variables to turn $\mathrm{d}x$ into $\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$, and the same for $\mathrm{d}y$ and $\mathrm{d}z$, this turns into the usual form of the vector line integral.

1.2 An example

Let's try this out. Let $\mathbf{F} = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$. Suppose that a particle moves from $(0,0,0)$ to $(1,1,1)$ along the curve $\mathbf{r}(t) = (t, t^2, t^3)$, where $t \in [0,1]$. What is the net work done by $\mathbf{F}$ on the particle?

Here, $\mathbf{F}(\mathbf{r}(t)) = t^3\,\mathbf{i} + t^2\,\mathbf{j} + t\,\mathbf{k}$ and $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$. Taking their dot product, we get

$$\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = t^3 \cdot 1 + t^2 \cdot 2t + t \cdot 3t^2 = t^3 + 2t^3 + 3t^3 = 6t^3.$$

Therefore the work done by $\mathbf{F}$ is

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{t=0}^1 6t^3\,\mathrm{d}t = 6\cdot \frac{t^4}{4}\bigg|_{t=0}^1 = \frac{6}{4} = \frac{3}{2}.$$

Let's try some variations. If we parameterize the same curve C by $\mathbf{r}(t) = (2t, 4t^2, 8t^3)$, where $t \in [0, \frac{1}{2}]$, then we describe a particle moving along the curve twice as fast. The value integral will not change! We'll get $\mathbf{F}(\mathbf{r}(t)) = 8t^3\,\mathbf{i} + 4t^2\,\mathbf{j} + 2t\,\mathbf{k}$, $\frac{\mathbf{dr}}{\mathbf{dt}} = 2\,\mathbf{i} + 8t\,\mathbf{j} + 24t^2\,\mathbf{k}$, and a dot product of $16t^3 + 32t^3 + 48t^3 = 96t^3$. The integral

$$\int_{t=0}^{1/2} 96t^3\,\mathrm{d}t$$

is still equal to $\frac{3}{2}$; in fact, we can obtain an integral equivalent to the first by the substitution $u = 2t$.

This is a general fact: **the vector line integral does not depend on the parameterization of the curve, provided that the orientation is respected**. If we reverse the orientation, for example by taking $\mathbf{r}(t) = (-t, t^2, -t^3)$ for $t \in [-1, 0]$, then the value of the integral will change sign and be $-\frac{3}{2}$, instead.

This particular example, for mysterious reasons we'll learn about later, has a further degree of symmetry. Suppose that we take a different path C' from $(0,0,0)$ to $(1,1,1)$: the straight-line path parameterized by $\mathbf{r}(t) = (t,t,t)$, where $t \in [0,1]$. Here,

$$\mathbf{F}(\mathbf{r}(t))\cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = (t\,\mathbf{i} + t\,\mathbf{j} + t\,\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k}) = 3t,$$

so the work done by $\mathbf{F}$ is

$$\int_{C'} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{t=0}^1 3t\,\mathrm{d}t = \frac{3}{2}t^2\bigg|_{t=0}^1 = \frac{3}{2}.$$

It is still the same!

This is **not** a universal property of the vector line integral, but it isn't just a coincidence either. It is something special to this particular vector field $\mathbf{F}$. In physics, there is a name for a force that obeys this property: such a force is called a **conservative force**. In mathematics, we sometimes call $\mathbf{F}$ a **conservative vector field**, though when we look at this property later, we'll have other names for it as well.

2 Other interpretations of the vector line integral

2.1 Flow and circulation integrals

Staying within the realm of physics for the moment, suppose that our vector field $\mathbf{F}$ is a velocity field: $\mathbf{F}(x,y)$ gives the velocity of some kind of fluid (such as air, or water) at the point (x,y). Can we give an interpretation to

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$$

in this setting?

In this context, the vector line integral is called a **flow integral**. The **flow** of **F** along C is another kind of wacky metric—you can tell because its units will be squared length over time. You can think of it as measuring a "net velocity" of how much our fluid follows the curve C, accumulated over the entire curve. (So the velocity is multiplied by a length.) Apparently,[2] this is useful when analyzing the aerodynamics of airplanes, where the difference between the flow of air along curves tracing the bottom versus the top of the wing is what tells us how much lift the airplane is getting.

As a special case, suppose C is a closed curve: it ends where it starts. (Remember, C is still an oriented curve, so in addition to drawing a loop, we decide to go either clockwise or counterclockwise around the loop.) Then the flow is called **circulation**, and measures the tendency of this fluid to follow this curve. If the curve C is going around a whirlpool, then we expect to get a high circulation (provided that the direction of C matches the direction of the whirlpool, of course; otherwise the circulation will be negative).

Borrowing from the velocity field setting, we tend to call the vector line integral a **circulation integral** in *all* cases where C is a closed curve—even if there's no fluid dynamics happening at all. Generally, the circulation integral measure the net tendency of the vector field **F** to loop around the curve C.

Let's look at an example. Take the two-dimensional vector field $\mathbf{F} = x\,\mathbf{i} + (x+y)\,\mathbf{j}$, and let C be the counterclockwise unit circle, with the usual parameterization $\mathbf{r}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. You can see both **F** and C displayed in Figure 1. Can we identify how much **F** is looping around the unit circle?

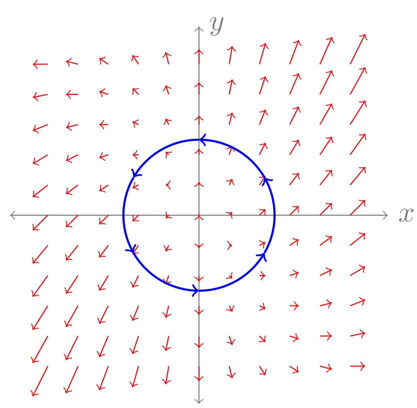

Figure 1: A circulation integral: how much is the vector field looping around the unit circle?

We have $\mathbf{F}(\mathbf{r}(t)) = \cos t\,\mathbf{i} + (\sin t + \cos t)\,\mathbf{j}$, and $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$. Taking the dot product of these two vectors, we get

$$-\sin t \cos t + (\sin t + \cos t)\cos t = \cos^2 t.$$

Now we integrate:

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{t=0}^{2\pi} \cos^2 t \,\mathrm{d}t = \int_{t=0}^{2\pi} \frac{1+\cos 2t}{2}\,\mathrm{d}t = \left(\frac{t}{2}\frac{\sin 2t}{4}\right)\Bigg|_{t=0}^{2\pi} = (\pi + 0) - (0 + 0) = \pi.$$

[2] I am not a physicist.

What does this *mean*?

Well, first of all, it suggests that overall, this vector field does have a tendency to loop counterclockwise around this circle.

How much of a tendency? Well, the circumference of this circle is 2π, so a circulation of π means that we're getting the same circulation as we would if our fluid constantly had a velocity of $\frac{1}{2}$ in the direction of the circle, at every point of C. That's not what's going on, of course: at some points such as $(0,1)$ or $(0,-1)$, the fluid is moving perpendicular to C, and at other points, it has mixed tendencies. The circulation is just the aggregate effect of what's going on at all those points at once.

2.2 Vector vs. scalar line integrals

Let's look at the two formulas side by side:

$$\int_C f(x,y,z)\,\mathrm{d}s = \int_{t=a}^{b} f(\mathbf{r}(t)) \left\| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right\| \mathrm{d}t \qquad \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{t=a}^{b} \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t.$$

We can express the second of these integrals in terms of the first, somewhat.

Let $\mathbf{T}(t)$ be the result of dividing $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ by its norm. This gives a unit vector tangent to C at the point $\mathbf{r}(t)$ (hence the awkward notation). In terms of this unit vector, we have

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{t=a}^{b} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{T}(t) \left\| \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right\| \mathrm{d}t.$$

(We divided by the norm, so we have to multiply by it again.) This is the scalar line integral of a quantity that depends in part on C itself: $f(\mathbf{r}(t)) = \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{T}(t)$. (This quantity is not defined for points not on C.) For this reason, you sometimes see a slightly different notation for the vector line integral:

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\,\mathrm{d}s.$$

Every once in a while, when $\mathbf{T}(t)$ has a very convenient form, this is actually helpful.

There's another way that the two types of integrals sometimes interact. Suppose that the curve C is actually a very boring curve: it is a line segment parallel to the x-axis, so that the y- and z-coordinates of points on C are all the same. In this case, the scalar line integral's $\mathrm{d}s$ (which intuitively represents a small change along C) can be replaced by $\mathrm{d}x$, and the scalar integral

$$\int_C f(x,y,z)\,\mathrm{d}x$$

that we get is *almost* the same as the vector line integral

$$\int_C f(x,y,z)\,\mathrm{d}x = \int_C f(x,y,z)\,\mathrm{d}x + 0\,\mathrm{d}y + 0\,\mathrm{d}z$$

of the vector field $\mathbf{F}(x,y,z) = f(x,y,z)\,\mathbf{i}$ along C.

I say *almost* the same, because one of these cares about orientation, and the other does not. The correspondence here works out provided that C is oriented in the direction of *increasing* x; otherwise, the scalar and vector line integrals have opposite signs!