Math 3272: Linear Programming[1] Mikhail Lavrov

Lecture 3: Review of linear algebra

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1 Systems of linear equations

In linear algebra, you learned how to solve a system of equations like this one:

Problem 1. *Solve for x and y:*

$$\begin{cases} 3x + y = 6 \\ 2x - y = -2 \end{cases}$$

In this lecture, I want to go over using Gaussian elimination to do this, and some finer points of the algorithm that we'll need to know for this class.

We begin by deciding that x will be the **basic variable** for the first equation. Having made this decision:

1. We scale the first equation so that the coefficient of x is 1. We get

$$\begin{cases} x + \frac{1}{3}y = 2 \\ 2x - \phantom{\frac{1}{3}}y = -1 \end{cases}$$

2. We subtract twice the first equation from the second, so that x is eliminated. (In general, we do this to eliminate the basic variable from every other equation.) We get

$$\begin{cases} x + \frac{1}{3}y = 2 \\ - \frac{5}{3}y = -5 \end{cases}$$

Next, we move on to the second equation. We pick a basic variable there as well; it can only be y, because that's the only variable contained in the equation. Again:

1. We scale the second equation so that the coefficient of y is 1. We get

$$\begin{cases} x + \frac{1}{3}y = 2 \\ \phantom{x + \frac{1}{3}}y = 3 \end{cases}$$

2. To clear y from the first equation, we subtract $\frac{1}{3}$ of the second equation. We get

$$\begin{cases} x = 1 \\ y = 3 \end{cases}$$

Now the solution can be read off directly: $x = 1$ and $y = 3$.

[1] This document comes from an archive of the Math 3272 course webpage: http://misha.fish/archive/3272-fall-2022

2 Infinitely many solutions

When the number of variables and the number of equations are equal, as in Problem 1, it is typical to get one solution at the end. Occasionally, other things might happen: an equation might end up simplifying to $0 = 0$ (in which case we get fewer constraints than expected) or to $0 = 1$ (in which case there is no solution at all). But these are rare.

In this class, we will almost exclusively deal with systems of equations which have more variables than constraints. In this case, it is guaranteed that there will either be **no** solutions or **infinitely many** solutions. We can still find the solutions by Gaussian elimination.

Let's look at an example.

Problem 2. *Describe all solutions to the system of equations*

$$\begin{cases} 2x_1 + x_2 - 2x_3 + x_4 = 0 \\ 3x_1 - x_2 + x_4 = 5 \end{cases}$$

Again, we begin by choosing x_1 as the basic variable in the first equation. Divide the first equation by 2, and then subtract 3 times the result from the second equation. We get:

$$\begin{cases} x_1 + \frac{1}{2}x_2 - x_3 + \frac{1}{2}x_4 = 0 \\ - \frac{5}{2}x_2 + 3x_3 - \frac{1}{2}x_4 = 5 \end{cases}$$

Next, we choose x_2 as the basic variable in the second equation. Multiply the second equation by $-\frac{2}{5}$ so that we get x_2 with coefficient 1. Then subtract $\frac{1}{2}$ of the result from the first equation. We get:

$$\begin{cases} x_1 - \frac{2}{5}x_3 + \frac{2}{5}x_4 = 1 \\ x_2 - \frac{6}{5}x_3 + \frac{1}{5}x_4 = -2 \end{cases}$$

Now Gaussian elimination is done: every equation has a basic variable which appears in that equation with coefficient 1, and nowhere else. But how do we get the solutions from this result?

We've gotten two **basic** variables x_1 and x_2 as well as two **non-basic** variables x_3 and x_4. If we just want one solution to the system of equations, we can set the non-basic variables to 0. Then the x_3 and x_4 terms disappear entirely, and our equations become $x_1 = 1$ and $x_2 = -2$. The resulting solution $(x_1, x_2, x_3, x_4) = (1, -2, 0, 0)$ is called a **basic solution**.

But Problem 2 asks for all solutions. To find these, we can set the non-basic variables to any value we want, and read off values for the basic variables. If we're going to be doing this a lot, it will help to move the non-basic variables to the other side:

$$\begin{cases} x_1 = 1 + \frac{2}{5}x_3 - \frac{2}{5}x_4 \\ x_2 = -2 + \frac{6}{5}x_3 - \frac{1}{5}x_4 \end{cases}$$

For example, if we plug in $x_3 = 5$ and $x_4 = 10$, we get $x_1 = 1 + \frac{2}{5}(5) - \frac{2}{5}(10) = -1$ and $x_2 = -2 + \frac{6}{5}(5) - \frac{1}{5}(10) = 2$: $(x_1, x_2, x_3, x_4) = (-1, 2, 5, 10)$ is also a solution.

3 Terminology and notation

In linear algebra, it is more common to say "pivot variables" and "free variables" instead of "basic variables" and "non-basic variables". In linear programming, the "basic" and "non-basic" are used almost exclusively, so we'll stick to that terminology.

When solving many of these equations by hand, it helps to find ways to write less. For example, we can write Problem 2 in matrix form as

$$\begin{bmatrix} 2 & 1 & -2 & 1 \\ 3 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

or in augmented matrix form as

$$\left[\begin{array}{cccc|c} 2 & 1 & -2 & 1 & 0 \\ 3 & -1 & 0 & 1 & 5 \end{array}\right]$$

I will not do this in these lecture notes, to stay consistent with the textbook's notation, but you are free to do so in assignments if you choose. I do recommend that you annotate the columns with the variables they correspond to, since this information will be important to track:

$$\begin{array}{rrrr|r} x_1 & x_2 & x_3 & x_4 & \\ \hline 2 & 1 & -2 & 1 & 0 \\ 3 & -1 & 0 & 1 & 5 \end{array}$$

When an equation has a basic variable, it helps to annotate that row with its basic variable. This is especially important when expressing the basic variables in terms of the non-basic variables, since that information does not exist anywhere else! For example,

$$\begin{cases} x_1 = 1 + \frac{2}{5}x_3 - \frac{2}{5}x_4 \\ x_2 = -2 + \frac{6}{5}x_3 - \frac{1}{5}x_4 \end{cases} \qquad \text{becomes} \qquad \begin{array}{c|r|rr} & & x_3 & x_4 \\ \hline x_1 & 1 & \frac{2}{5} & -\frac{2}{5} \\ x_2 & -2 & \frac{6}{5} & -\frac{1}{5} \end{array}$$

4 Choosing a different basis

In linear algebra, when all we wanted to do was solve the system of linear equations, it did not really matter which variables were chosen to be the basic variables: any choice that worked was equally good. The simplex method, as we'll see in the next lecture, relies on moving between different choices of basic variables.

Let's take another look at the system of equations from Problem 2, but with a different goal.

Problem 3. *Parameterize all solutions to the system of equations*

$$\begin{cases} 2x_1 + x_2 - 2x_3 + x_4 = 0 \\ 3x_1 - x_2 \phantom{{}- 2x_3} + x_4 = 5 \end{cases}$$

by expressing x_2 and x_3 in terms of x_1 and x_4.

There are three ways to solve this problem, and we will look at them all.

4.1 Solving the problem from scratch

We can continue using our previous method, but choose our basic and nonbasic variables differently. Since we want x_2 and x_3 in terms of x_1 and x_4, we want x_2, x_3 to be our basic variables and x_1, x_4 to be our nonbasic variables.

Begin by choosing x_2 as the basic variable in the first equation. (Choosing x_2 first rather than x_3 is an arbitrary decision.) We don't need to do any division, and we should just add the first equation to the second to eliminate x_2:

$$\begin{cases} 2x_1 + x_2 - 2x_3 + x_4 = 0 \\ 5x_1 \phantom{{}+x_2} - 2x_3 + 2x_4 = 5 \end{cases}$$

Next, choose x_3 as the basic variable in the second equation. We should divide by -2, and then add twice what we get from the first equation. (Equivalently, subtract the second equation from the first, then divide the second equation by -2.) We get:

$$\begin{cases} -3x_1 + x_2 \phantom{{}+x_3} - x_4 = -5 \\ -\frac{5}{2}x_1 \phantom{{}+x_2} + x_3 - x_4 = -\frac{5}{2} \end{cases}$$

To read off x_2, x_3 in terms of x_1, x_4, we can move those terms to the other side, getting

$$\begin{cases} x_2 = -5 + 3x_1 + x_4 \\ x_3 = -\frac{5}{2} + \frac{5}{2}x_1 + x_4 \end{cases}$$

4.2 Modify an existing solution

The approach above is fine if we saw Problem 3 first, but since we solved Problem 2 already, it seems a shame to ignore all that effort. Here is that solution again:

$$\begin{cases} x_1 = 1 + \frac{2}{5}x_3 - \frac{2}{5}x_4 \\ x_2 = -2 + \frac{6}{5}x_3 - \frac{1}{5}x_4 \end{cases}$$

To minimize effort, we can take this solution as a starting point. Let's begin by eliminating x_3 from the second equation. To do this, just subtract 3 times the first equation:

$$\begin{cases} x_1 = 1 + \frac{2}{5}x_3 - \frac{2}{5}x_4 \\ x_2 - 3x_1 = -5 \phantom{{}+\frac{2}{5}x_3} + \phantom{\frac{2}{5}}x_4 \end{cases}$$

We want x_3 on the left-hand side and x_1 on the right-hand side, so just move those terms (in both equations)

$$\begin{cases} - \frac{2}{5}x_3 = 1 - x_1 - \frac{2}{5}x_4 \\ x_2 \phantom{{}-\frac{2}{5}x_3} = -5 + 3x_1 + \phantom{\frac{2}{5}}x_4 \end{cases}$$

Finally, multiply the first equation by $-\frac{5}{2}$ so that x_3 appears with a coefficient of 1 on the left-hand side:

$$\begin{cases} x_3 = -\frac{5}{2} + \frac{5}{2}x_1 + x_4 \\ x_2 = -5 + 3x_1 + x_4 \end{cases}$$

In this example, with only two equations, this does not seem like more effort than solving from scratch. This approach (which we'll call **pivoting** in the future) shines if we have many equations, and we are only making a minor change to the set of basic variables.

4.3 Multiply by an inverse matrix

The final method we'll look at will not be relevant for a while, but it's an interesting trick. Start with the matrix form of the system of equations:

$$\begin{bmatrix} 2 & 1 & -2 & 1 \\ 3 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

We want to solve for x_2 and x_3, so just take the second and third columns of the coefficient matrix on the left:

$$\begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$$

To get a system of equations in which x_2 and x_3 are the basic variables, find the inverse of this matrix:

$$\begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}^{-1} = \frac{1}{1 \cdot 0 - (-2) \cdot (-1)} \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Then, left-multiply both sides of the matrix equation by that inverse:

$$\begin{bmatrix} 0 & -1 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 & 1 \\ 3 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

This simplifies to

$$\begin{bmatrix} -3 & 1 & 0 & -1 \\ -\frac{5}{2} & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5 & -\frac{5}{2} \end{bmatrix}$$

and now we directly have the row-reduced form of the system of equations. Moving from matrices back to equations, this result tells us that:

$$\begin{cases} -3x_1 + x_2 \qquad\quad - x_4 = -5 \\ -\frac{5}{2}x_1 \qquad + x_3 - x_4 = -\frac{5}{2} \end{cases}$$

All that's left is to isolate x_2 in the first equation and x_3 in the second, and we'll have the same solution we've found twice already:

$$\begin{cases} x_2 = -5 + 3x_1 + x_4 \\ x_3 = -\frac{5}{2} + \frac{5}{2}x_1 + x_4 \end{cases}$$