Lecture 15: More topics related to matchings

Note: the proof of the Tutte–Berge formula in today's lecture notes is included for completeness, but we will probably not cover it in class. I think this proof is still valuable to read on your own; let me know if you have questions.

1 Applications of König's theorem

1.1 Hall's theorem

We ended the previous lecture by proving the following theorem:

Theorem 1.1 (König)**.** *In a bipartite graph G, the number of edges in a largest matching is equal to the number of vertices in a smallest vertex cover.*

König's theorem is good for upper bounds on the size of a matching, but occasionally we just want to verify that a matching which is as large as possible exists. For this, we have Hall's theorem.

For a set of vertices S in a graph G, let $N(S)$ be the set of all vertices adjacent to a vertex in S: the **neighborhood** of S. In some cases, we might be simultaneously discussing multiple graphs with with vertex sets containing S; in that case, we write $N_G(S)$ to specify the graph G we're talking about.

Theorem 1.2 (Hall)**.** *A bipartite graph G with bipartition (A, B) has a matching that covers all of A if and only if it has the following property (known as* ***Hall's condition****):*

$$\text{For all } S \subseteq A,\ |N(S)| \geq |S|.$$

In particular, if $|A| = |B|$, this is the condition for G to have a perfect matching.

Proof. Where does Hall's condition come from? The logic is: if every element of S is covered by a matching, each vertex $u \in S$ is matched with some vertex $v \in N(S)$. (By "matched with" I just mean that edge uv is in the matching.) So for the $|S|$ different vertices in S, we need at least $|S|$ different vertices in $N(S)$ for them to be matched with. This condition generalizes some special cases:

- When $|S| = 1$, it says that in order for a matching to cover A, every vertex must have a neighbor.
- Checking the condition for $S = \{u, v\}$ rules out the case where u and v both have only one neighbor, and it's the same vertex.
- When $S = A$, Hall's condition implies that we must have $|B| \geq |A|$ to get a matching that covers all of A.

[1] This document comes from an archive of the Math 3322 course webpage: http://misha.fish/archive/3322-fall-2024

But this is all arguing that Hall's condition is necessary.

To prove it's sufficient, we will use König's theorem. Specifically, we will assume that there is no matching that covers all of A. Then, we will find a subset $A \subseteq S$ that violates Hall's condition.

Another way to say "there is no matching that covers all of A" is "there is no matching of size $|A|$", because every edge of a matching covers exactly one vertex in A. If there is no matching of size $|A|$, then by König's theorem, there is a vertex cover U of size at most $|A| - 1$.

Now let $S = A \setminus U$: the set of all vertices in A that are not part of the vertex cover U. If $v \in S$ and w is adjacent to v, then to cover edge vw, we must have $w \in U$ (since we cannot have $v \in U$). Therefore all of $N(S)$ must be contained in U.

Therefore U contains at least the $|A| - |S|$ vertices in A but not in U, and also at least the $|N(S)|$ vertices in $N(S)$ (which are all in B, so we're not double-counting). But we started by assuming that $|U| \leq |A| - 1$. Therefore

$$\big(|A| - |S|\big) + |N(S)| \leq |A| - 1$$

which can be rearranged to $|N(S)| \leq |S| - 1$, proving that S violates Hall's condition. □

1.2 Regular bipartite graphs

There's lots of applications of Hall's theorem to various settings. Sometimes, we find a small set that violates Hall's condition, and immediately conclude a perfect matching is impossible. Sometimes we use it as a lemma for other theorems.

For example, there's the following (coincidentally also proved by König, so let's just call it "Theorem 1.3"):

Theorem 1.3. *If G is an r-regular bipartite graph (every vertex has degree r), then it has a perfect matching.*

Proof. Just to get some intuition, here is an observation first. If G has a bipartition (A, B), then we can count the edges of G in two ways. First, we could sum the degrees of all vertices in A and get $r|A|$: this counts every edge once, because every edge has an endpoint in A. We could also sum the degrees of all vertices in B and get $r|B|$. The two methods must give the same answer, so $r|A| = r|B|$, and $|A| = |B|$.

So there is always *hope* for a perfect matching in a regular bipartite graph, and this theorem goes on to say that the perfect matching always exists.

We will prove the theorem two ways: once using Hall's theorem, and once using König's theorem.

Using Hall's theorem. We will pick an arbitrary set $S \subseteq A$, and check that it satisfies Hall's condition.

There are exactly $r|S|$ edges with one endpoint in S. Each one of them has its other endpoint in $N(S)$. This doesn't mean that there are $r|S|$ vertices in $N(S)$: the same vertex in $N(S)$ could be the endpoint of many of the $r|S|$ edges. But actually, it can be the endpoint of at most r of them: so $|N(S)| \geq \frac{r|S|}{r} = |S|$.

Putting it differently: we have

$$\sum_{v \in S} \deg(v) \le \sum_{w \in N(S)} \deg(w)$$

because the first sum counts all edges between S and $N(S)$, and the second sum counts all edges incident to a vertex in $N(S)$; this includes all edges counted by the first sum, and maybe more. But since all degrees are equal to r, we get $r|S| \le r|N(S)|$, so $|N(S)| \ge |S|$.

Either way, proving this and applying Hall's theorem gets us a perfect matching.

Using König's theorem. If $|A| = |B| = n$, then there are rn total edges in the graph. Let U be a minimum vertex cover. Each vertex of U can cover at most r edges, so to cover all rn edges, we need $|U| \ge n$. Also, $|U| = n$ is achievable: just take $U = A$, or $U = B$, for example.

Since the minimum vertex cover has n vertices, by König's theorem, the maximum matchng has n edges. This must cover all $2n$ vertices in the graph, so it is a perfect matching. □

Theorem 1.3 generalizes to bipartite graphs that are "biregular". A (r, s)**-biregular bipartite graph** is a bipartite graph which has a bipartition (A, B) such that every vertex in A has degree r, and every vertex in B has degree s.

By double-counting as we did earlier, we have $r|A| = s|B|$, so if $r \ne s$, then $|A| \ne |B|$. We can't hope for a perfect matching. However, we can show that there is a matching of size at least $\min\{|A|, |B|\}$: a matching that covers the smaller side of the bipartition.

2 Generalizing to non-bipartite graphs

When G is not a bipartite graph, our previous results and algorithms do not apply. In particular, though vertex covers still give upper bounds on the size of a maximum matching, they do not necessarily give *useful* upper bounds. So what can we still do?

2.1 Augmenting paths

Augmenting paths still help us!

Theorem 2.1. *If M is a matching in any graph G, then either M is a maximum matching, or G has an M-augmenting path.*

Proof. Suppose M is a matching in G, but it is not a maximum matching: there is a matching N which is larger. We will look at the symmetric difference $M \oplus N$.

In this symmetric difference, every vertex has degree 0, 1, or 2 (every vertex has up to 1 neighbor from M, and up to 1 neighbor in N). In a graph where every degree is at most 2, we can see the following kinds of components:

- Isolated vertices, which have degree 0.

  These can appear in $M \oplus N$ in two ways: as vertices not covered by either matching, and as vertices covered by both matchings using the same edge.

- Cycles, in which every vertex has degree 2.

  In the particular case of $M \oplus N$, these must be even cycles; that's because two consecutive edges of a cycle in $M \oplus N$ cannot come from the same matching, so they must alternate. As a consequence, each cycle has an equal number of edges from M and N.

- Paths, in which the first and last vertex have degree 1, and intermediate vertices (if there are any) have degree 2.

  In these paths, we alternate between "edge only in M" and "edge only in N". We are hoping to find a path which begins and ends with an edge of N: this would be an M-augmenting path.

  But we could also see paths which begin with an edge of one matching, and end with an edge of the other: these have an equal number of edges from M and N. We could even see N-augmenting paths, with more edges from M than from N.

Because $|M| < |N|$, we know that $M \oplus N$ must contain a component with more edges from N than from M. The only such component we've found is an M-augmenting path, so in particular an M-augmenting path must exist. □

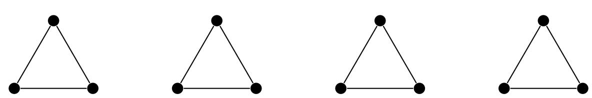
The problem is that we cannot easily *find* an M-augmenting path; the algorithm we used before only works for bipartite graphs. There are algorithms that do this, but they are much more complicated.

2.2 Odd fragments

Let's see what kind of obstacles we have that limit the size of a matching in a general graph, in addition to vertex covers.

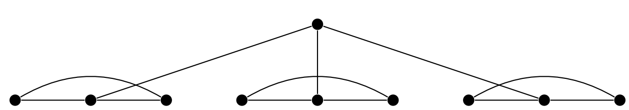
One possible obstacle is parity. In the complete graph K_{99}, the largest vertex cover is very large, and tells us nothing useful. But we know that the largest matching has 49 edges (covering only 98 vertices), because 50 edges would require 100 vertices. In general, in any graph with an odd number of vertices, every matching must leave out at least one vertex.

It's not just the total number of vertices that causes problems. Suppose we have a 12-vertex graph that consists of 4 copies of K_3 (this graph is often called $4K_3$ for short):



The matching number $\alpha'(4K_3)$ is only 4, because we can pick at most one edge from each component. It doesn't matter that the total number of vertices is even.

Finally, we can see parity becoming an issue even in connected graphs. Consider the graph below, with three copies of K_3 joined together by a 10^{th} vertex:



On its own, each copy of K_3 can only support one edge of a matching; a vertex will be left over. The 10^{th} vertex can rescue and pick up one of those leftover vertices—but only one. As a result, the matching number of this graph is 4.

Let $o(H)$ denote[2] the number of odd components in a graph H. Then, suppose there is a subset $U \subseteq V(G)$ such that $o(G-U)$ is very large. Then at least $o(G-U)-|U|$ vertices are left uncovered by any matching: each matching misses a vertex in each odd component of $G-U$, except for $|U|$ vertices "saved" by U. So at most $|V(G)| - o(G-U) + |U|$ vertices are covered.

The size of a matching is half the number of colored vertices. So the parity argument gives us the upper bound

$$\alpha'(G) \le \frac{1}{2}\left(|V(G)| - o(G-U) + |U|\right)$$

for any $U \subseteq V(G)$.

The striking fact is that there is always some set U for which this upper bound is exactly the size of a maximum matching! In other words, the obstacle we've outlined is really the only obstacle to matching in any graph (just like vertex covers are the only obstacle in bipartite graphs).

Theorem 2.2 (Tutte–Berge formula)**.** *For all graphs G, $\alpha'(G)$ is equal to the minimum of the quantity*

$$\frac{1}{2}\left(|V(G)| - o(G-U) + |U|\right)$$

over all $U \subseteq V(G)$.

Proof. We will induct on $|V(G)|$, but this induction is a bit unusual. We use induction only to simplify "easy" cases, until we end up at a specific kind of "hard" case.

One "easy case" is when G is not connected. If G has connected components $G_1, G_2, \dots, G_k$, then we can apply the induction hypothesis to each component, finding a $U_i \subseteq V(G_i)$ for each i. Because $\alpha'(G) = \alpha'(G_1) + \cdots + \alpha'(G_k)$, and each $\alpha'(G_i)$ is equal to $\frac{1}{2}(|V(G_i)| - o(G_i - U_i) + |U_i|)$, the set $U = U_1 \cup U_2 \cup \cdots \cup U_k$ turns out to be the set we want to find in the theorem.

Another "easy case" is when G has a vertex v that is present in *every* maximum matching. (In particular, this includes graph G with a perfect matching: then we can choose any vertex to be v.) We apply the induction hypothesis to $G - v$, finding $U' \subseteq V(G-v)$ such that

$$\alpha'(G-v) = \frac{1}{2}\left(|V(G-v)| - o(G-v-U') + |U'|\right).$$

Since v is in every maximum matching of G, we must have $\alpha'(G-v) = \alpha'(G) - 1$. We can check that this means $U = U' \cup \{v\}$ satisfies the theorem.

So now we are left with the case of a connected graph G in which every vertex is left uncovered by some maximum matching. This case is the most technical, and we will not use induction here. We will prove that G has an odd number of vertices, and has a matching covering all but

[2] If it bothers you that for once, the graph theorists have decided to use a normal, non-Greek letter for a graph parameter, feel free to imagine that the o in $o(H)$ is a lowercase "omicron". In any case, this is not notation we will need in future lectures.

one of them. Then taking $U = \varnothing$ will satisfy the theorem: we have $o(G - U) = o(G) = 1$, so $\frac{1}{2}(|V(G)| - o(G - U) + |U|) = \frac{1}{2}(|V(G)| - 1) = \alpha'(G)$.

To prove this, assume the opposite, for the sake of contradiction: that there is a maximum matching M that does not cover two vertices x, y. Out of all choices of M, x, y, let's pick one where $d(x, y)$ is as small as possible.

Even so, $d(x, y)$ cannot be 1: then xy would be an edge, and $M \cup \{xy\}$ would be a larger matching. So $d(x, y) \geq 2$. What does this get us? It means we can choose z on some shortest $u - v$ path. Then $d(x, y) = d(x, z) + d(z, y)$, so $d(x, z)$ and $d(z, y)$ are both smaller than $d(x, y)$.

By the case, there is another maximum matching N that does not cover z. Then every such N covers x (otherwise N, x, z would have been chosen in place of M, x, y) and y (for the same reason).

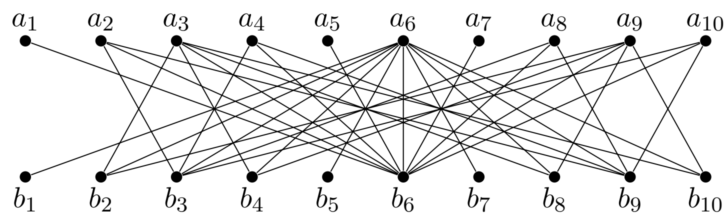
What does the symmetric difference $M \oplus N$ look like? It cannot contain an M-augmenting or N-augmenting path, since both M and N are maximum matchings. So it contains some even cycles, and also some paths that start at a vertex covered by M and not N, and end at a vertex covered by N and not M.

In particular, $M \oplus N$ must contain an $x - x'$ path P_x and a $y - y'$ path P_y, where x', y' are two vertices covered by M and not by N. If $x' \neq z$, then $N \oplus P_x$ is a maximum matching that leaves both x and z uncovered, so $N \oplus P_x, x, z$ would have been chosen in place of M, x, y. If $x' = z$, then $y' \neq z$, and by the same reasoning, $N \oplus P_y, y, z$ would have been chosen in place of y.

We get a contradiction, so the assumption (the existence of two vertices x, y uncovered by a maximum matching M) must be invalid. Therefore there is a matching covering all but one vertex of G, and we are done. □

3 Practice problems

1. The multiples-of-six graph from Lecture 13 is shown again below:



   Prove that it does not have a perfect matching, but this time, using Hall's theorem.

2. The Tutte–Berge's formula gets its name because it is a generalization (due to Berge) of a theorem known as Tutte's theorem, which gives the condition for when a general graph has a perfect matching.

   Tutte's theorem can be stated as follows. "A graph G has a perfect matching if and only if it has the following property (known as **Tutte's condition**): for all $U \subseteq V(G)$, ________."

   Use Theorem 2.2 to fill in the blank.

3. Let G be a bipartite graph with n vertices on each side of the bipartition whose minimum degree $\delta(G)$ is greater than $n/2$.

   Prove that G has a perfect matching.

4. We know that if G is a r-regular bipartite graph, then it has a perfect matching.

   Prove that, actually, G has r perfect matchings $M_1, M_2, \dots, M_r$ that partition $E(G)$: every edge of G is in exactly one of the matchings.

5. Let $G_{n,k}$, where $0 \le k \le n-1$, be the bipartite graph defined as follows: its vertices are all the subsets of $\{1, 2, \dots, n\}$ of size k or $k+1$, and it has an edge vw if $v \subset w$. (That is, v must be a set of size k, w must be a set of size $k+1$, and we must be able to add a single element to v to get w.)

   (a) Explain how $G_{n,k}$ is an induced subgraph of the hypercube graph Q_n.

   (b) Prove that $G_{n,k}$ has a matching that covers all the vertices on the side with fewer vertices.

   (c) A *chain* of sets is a sequence of nested sets $A_1 \subset A_2 \subset \cdots \subset A_k$. Prove that the 2^n subsets of $\{1, 2, \dots, n\}$ can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ chains.

   (Hint: combine the matchings in $G_{n,0}$ through $G_{n,n-1}$).

6. Theorem 1.3 implies that every 3-regular bipartite graph has a perfect matching.

   Prove that the word "bipartite" cannot be left out: give an example of a 3-regular graph which is *not* bipartite, and does *not* have a perfect matching.

7. For bipartite graphs, the Tutte–Berge formula (Theorem 2.2) becomes equivalent to König's theorem (Theorem 1.1), though this is not immediately obvious.

   (a) Prove that when G is bipartite and $U \subseteq V(G)$ is a vertex cover, the quantity

   $$\frac{1}{2}\left(|V(G)| - o(G-U) + |U|\right)$$

   (from the Tutte–Berge formula) simplifies to just $|U|$.

   (b) Prove that when G is bipartite and $U \subseteq V(G)$ is any set, we can extend it to a vertex cover U' (that is, with $U \subseteq U'$) such that

   $$\frac{1}{2}\left(|V(G)| - o(G-U) + |U|\right) \geq \frac{1}{2}\left(|V(G)| - o(G-U') + |U'|\right).$$

   (By part (a), the right-hand side of this inequality is equal to $|U'|$.)

   (c) Explain why these two facts tell us that when G is bipartite, the Tutte–Berge theorem implies König's theorem and vice versa.