Math 3322: Graph Theory[1]	Mikhail Lavrov

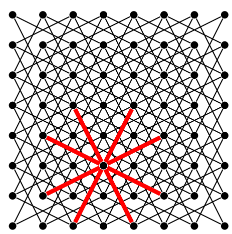
Lecture 18: Hamiltonian cycles

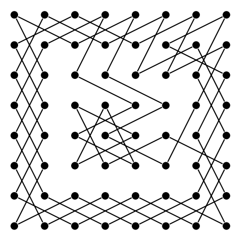
October 17, 2024 — *Kennesaw State University*

1 Some examples

1.1 Knight's tours

A knight is a chess piece that (on the 8×8 chessboard) moves by taking two steps in one direction, turning 90°, and taking one more step. We are all graph theorists here, and so for us this means that a knight moves along the edges of the 64-vertex graph in the first diagram below. (For clarity, I have highlighted the 8 edges out of the typical vertex.)



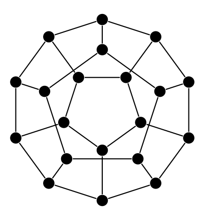


The oldest Hamiltonian cycle problem in history is finding a **closed knight's tour** of the chessboard: the knight must make 64 moves to visit each square once and return to the start. That's exactly a Hamiltonian cycle in the graph we just drew. One solution is shown in the second diagram above.

We will not try to solve the 8×8 problem today. However, we will think about the problem for some smaller boards, and see what happens there!

1.2 The Icosian game

The problem of finding Hamiltonian cycles is named after William Rowan Hamilton, an Irish mathematician who was a big fan of the dodecahedron (the 3D shape whose sides are twelve pentagons). The graph below is the "skeleton graph" of the dodecahedron: the graph whose vertices are the corner points of the dodecahedron, and whose edges are the literal physical edges between those corner points.



[1]This document comes from an archive of the Math 3322 course webpage: http://misha.fish/archive/3322-fall-2024

Hamilton's "Icosian[2] game" was a puzzle to find a Hamiltonian cycle of this graph. The graph is small enough that we can try to do this by hand without fancy theorems. Usually the following approach will work:

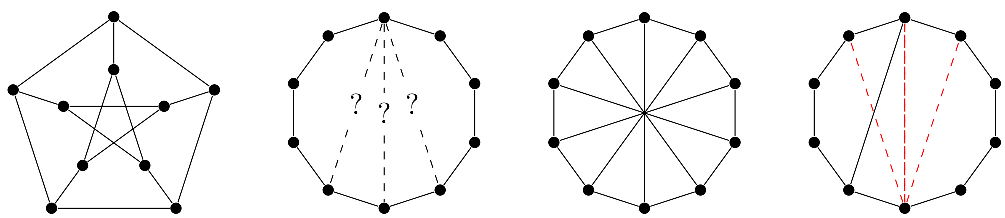
1. Draw a few edges of the cycle as the fancy strikes you. (It's possible to make a mistake here, but unlikely.)
2. Every vertex has degree 2 in the cycle, but degree 3 in this graph. So if we've decided that edges uv and vw are part of the cycle, then the third edge out of v cannot be part of the cycle; get rid of it.
3. Similarly, if we've already eliminated one edge out of a vertex, it means that the cycle must use the other two edges through that vertex.
4. Finally, we should avoid edges that would result in the cycle looping back on itself too early (before it visits all 20 vertices).

If steps 2–4 don't give us enough information to solve for the rest of the cycle, then we can guess again.

As you can see, this is not a systematic approach: there's lots of guessing involved! The truth is that there's no systematic approach to finding Hamiltonian cycles that doesn't use brute force.

1.3 The Petersen graph

Does the Petersen graph (first diagram below) have a Hamiltonian cycle?



There's a way of thinking about this question which will be useful to us in future arguments. If the Petersen graph has a Hamiltonian cycle, then it should be possible to find a different drawing of the same graph in which all 10 vertices are arranged around the boundary of a circle, with adjacent edges connected (second diagram above).

The Petersen graph is 3-regular, so each vertex has a third edge out of it. But we don't know which edges to draw yet because we don't know if such a diagram exists: that depends on whether the Petersen graph has a Hamiltonian cycle or not.

The Petersen graph has an unusual property: it has no cycles of length 3 or 4. (Check this! It's enough to check that none of the neighbors of a vertex are connected, and that no two vertices share a neighbor.) So our hypothetical second diagram of the Petersen graph must also have this property.

[2] It might be confusing why it's called "icosian" when it's played on the dodecahedron; the icosahedron is a different shape with 20 triangular faces. Well, "icosa-" is the Greek prefix that means 20, and the graph we're looking at has 20 vertices.

In particular, from each vertex, there are only three possible candidates from the third edge: it must go either to the opposite vertex or to its neighbors. (Also shown in the second diagram.) Otherwise, we get a cycle of length 3 or 4.

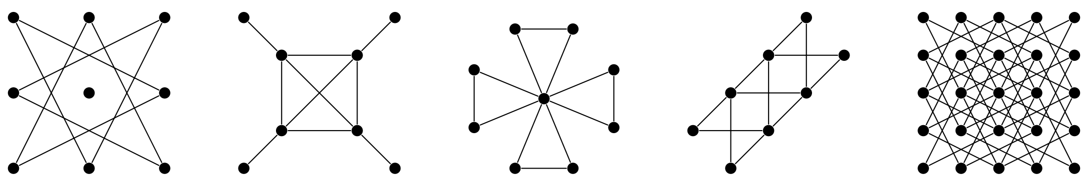
It does not work to connect every vertex to its opposite. That gets us the third diagram, which definitely has cycles of length 4. So there's some vertex v that's *not* connected to its opposite vertex w. This is shown in the fourth diagram above, where v is the top vertex and w is the bottom vertex. But now, all three options for w's third edge are bad: it cannot be v (that would make v have degree 4) and it cannot be either of v's neighbors (that would create a cycle of length 4).

So actually, the diagram we want cannot exist, and the Petersen graph is not Hamiltonian!

2 Necessary conditions

The argument for the Petersen graph just now was long and complicated. Fortunately, in many cases, we can rule out a Hamiltonian cycle much more easily. Consider the following examples; some of these come from other situations we've looked at in class, and others are knight's graphs for smaller chessboards.

I recommend stopping to think of a reason for each of these graphs why it is not Hamiltonian.



Here's what we can say:

1. In the first graph (the 3×3 knight's graph) there is no way to get to the middle vertex, so there can't be a Hamiltonian cycle. A general principle: a graph that is not connected cannot be Hamiltonian.

2. In the second graph, some vertices have degree 1. There is no way a cycle can both enter and exit those vertices. A general principle: a Hamiltonian graph must have minimum degree at least 2.

3. In the third graph, there is a cut vertex: a vertex we can delete to disconnect the graph.[3] Hamiltonian graphs cannot have cut vertices, because a closed walk would need to pass through the cut vertex multiple times to visit every vertex.

4. In the fourth graph, some vertices have degree 2, which is not a problem by itself. However, if a vertex has degree 2, both of its edges must be part of every Hamiltonian cycle. And if we select both edges out of every degree-2 vertex in this graph, we end up with two 4-cycles instead!

5. The fifth graph (the 5×5 knight's graph) is bipartite, as is any other knight's graph. In the usual black-and-white coloring of a chessboard, every move takes the knight to a square of

[3] We'll learn more about cut vertices later in the semester.

a different color. In particular, along a Hamiltonian cycle, the colors should alternate black and white.

However, the 5×5 chessboard has 12 squares of one color and 13 of the other. These can't alternate black and white, because there's not an equal number of them.

A general principle: a bipartite graph in which the two parts are different sizes cannot be Hamiltonian.

Surprisingly, all five arguments are special cases of one result!

We say that a graph G is **tough** if, for any $k \geq 1$, deleting k vertices from G results in at most k connected components.

Lemma 2.1. *For all $n \geq 3$, the cycle graph C_n is tough.*

Proof. If we delete any k vertices from C_n, we get a graph with $n - k$ vertices and no cycles: an $(n - k)$-vertex forest. Each deleted vertex has degree 2 in C_n, so it costs us at most two edges: at most, because we might delete both endpoints of an edge. Therefore the forest has at least $n - 2k$ edges.

In a forest, the number of components is the number of vertices minus the number of edges; in this case, this is at most $(n - k) - (n - 2k) = k$. □

(You can also give a more specific proof of Lemma 2.1 that analyzes the "shape" of the components you get. I include this proof because it's a nice review of properties of forests, and a glimpse of how they can be used to help us in other scenarios.)

Corollary 2.2. *All Hamiltonian graphs are tough.*

Proof. Every n-vertex Hamiltonian graph G contains a copy of C_n as a subgraph: that's what a Hamiltonian cycle is! Deleting k vertices from G will leave that subgraph in at most k pieces, by Lemma 2.1. It's possible that G has some extra edges not in this subgraph, but those cannot increase the number of connected components. □

In each of the examples at the beginning of this section, we can find a set of k vertices whose deletion leaves more than k components. (See if you can do this in all cases.) Therefore they are not tough—and therefore they are not Hamiltonian.

This is a necessary condition: to be Hamiltonian, a graph must be tough. However, it is not sufficient. It takes some casework, but we can check that the Petersen graph is tough, even though (as we saw earlier) it is not Hamiltonian.

3 Sufficient conditions

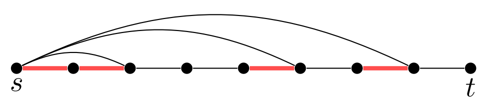
The prettiest condition that *does* guarantee Hamiltonian cycles in a graph comes from the Bondy–Chvátal theorem, which I like to call the "The magic was inside you all along" theorem.

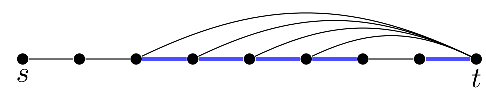
Theorem 3.1 (Bondy–Chvátal)**.** *In a graph G with n vertices, suppose s and t are two non-adjacent vertices with* $\deg(s) + \deg(t) \geq n$*.*

If $G + st$ (the graph we obtain from G by adding edge st) has a Hamiltonian cycle, then so does G.

Proof. Take a Hamiltonian cycle in $G + st$. If edge st is not part of that cycle, we're done: it also exists in G. If edge st *is* part of that cycle, then deleting that edge leaves an $s - t$ path in G that visits every single vertex: $(s = v_1, v_2, \ldots, v_{n-1}, v_n = t)$

Moreover, because $\deg(s) + \deg(t) \geq n$, there must be many other edges from s or t in the graph. Let's take a closer look at those edges.

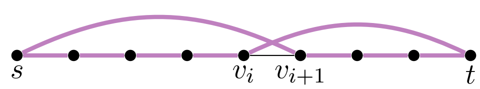




Let S be the set of edges $v_i v_{i+1}$ on the path in which v_{i+1} (the endpoint closer to t) is adjacent to s. These are highlighted in red in the first diagram above (for an example set of edges out of s). Because we get one edge in S for every neighbor of s, we know that $|S| = \deg(s)$.

Similarly, let T be the edges $v_i v_{i+1}$ on the path in which v_{i+1} (the endpoint closer to s) is adjacent to t. These are highlighted in blue in the second diagram; as before, $|T| = \deg(t)$.

Because $|S| + |T| = \deg(s) + \deg(t) \geq n$, but the path only has $n - 1$ edges, there must be overlap! Some edge $v_i v_{i+1}$ must be both in S and in T.



In this case, $(s = v_1, v_2, \ldots, v_i, t = v_n, v_{n-1}, \ldots, v_{i+1}, s)$ is a Hamiltonian cycle. $\square$

Essentially, the Bondy–Chvátal theorem says that edges like st don't help create a Hamiltonian cycle if one wasn't already there.

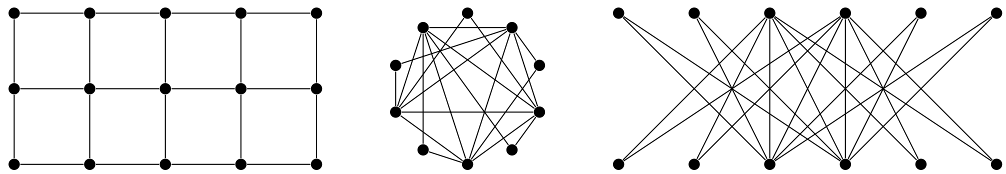
To apply this theorem, we use the following logic. Take any graph G, then add every single edge that the theorem applies to. (The resulting graph is called the **Bondy–Chvátal closure** of G.) If the result is Hamiltonian, then the original graph is also Hamiltonian. However, it might be easier to find a Hamiltonian cycle in the new graph, because it has more edges to work with.

Corollary 3.2 (Dirac's theorem)**.** *If G has $n \geq 3$ vertices and minimum degree $\delta(G) \geq \frac{1}{2}n$, then G is Hamiltonian.*

Proof. Here, if we take the closure of G, we end up with K_n: for any two non-adjacent vertices s and t, $\deg(s) + \deg(t) \geq \frac{1}{2}n + \frac{1}{2}n = n$. Since K_n is definitely Hamiltonian, G must also be Hamiltonian. $\square$

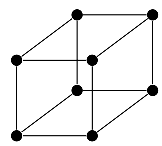
4 Practice problems

1. For each of the graphs below, find a Hamiltonian cycle, or explain why a Hamiltonian cycle cannot exist.



2. Determine the values of n for which the graph $\overline{C}_n$ (the complement of the cycle graph on n vertices) is Hamiltonian.

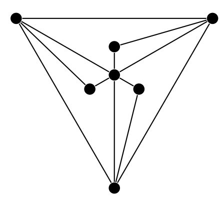
3. (a) Find a Hamiltonian cycle in the cube graph Q_3:



   (b) Use induction to prove that the hypercube graph Q_n has a Hamiltonian cycle for all $n \geq 3$.

   (Hint: the induction must use a Hamiltonian cycle in Q_{n-1} to help construct a Hamiltonian cycle in Q_n. Can you think of a way to use the Hamiltonian cycle in Q_2 to help with finding the Hamiltonian cycle you found in Q_3?)

4. In Chvátal's original paper defining tough graphs, the following example is given:



   (a) Prove that this graph is not Hamiltonian.

   (b) Prove that this graph is tough.

5. (a) Prove that any graph with the degree sequence $2, 5, 5, 7, 7, 7, 7, 7, 7$ is Hamiltonian.

   (b) State a general condition on degree sequences $d_1, d_2, \ldots, d_n$ which guarantees that any graph with that degree sequence is Hamiltonian, and generalizes your argument from part (a).

6. Let G be a bipartite graph with bipartition $V(G) = A \cup B$. Prove that if there is a subset $S \subseteq A$ with $|N(S)| < |S|$, then G cannot be Hamiltonian.

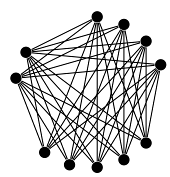
7. A graph is called **traceable** if it has a **Hamiltonian path**: a path that visits every vertex exactly once (but does not return to where it started). Many of the properties that let us see if a graph is Hamiltonian can be modified to help us determine if a graph is traceable.

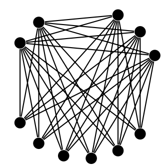
(a) Prove the following generalization of Corollary 2.2: if a graph G is traceable, then for any $k \geq 0$, deleting k vertices from G results in a graph with at most $k+1$ connected components.

(b) Prove the following generalization of Dirac's theorem: if a graph G has minimum degree $\delta(G) \geq \frac{1}{2}(n-1)$, then G is traceable.

8. A complete tripartite graph is formed by taking three groups of vertices A, B, and C, then adding an edge between every pair of vertices in **different** groups. We write $K_{a,b,c}$ for the complete tripartite graph with $|A| = a$, $|B| = b$, and $|C| = c$.

For example, below are diagrams of $K_{2,4,5}$ (left) and $K_{2,3,6}$ (right).





(a) One of these graphs is Hamiltonian. Find a Hamiltonian cycle in that graph.

(b) The other of these graphs is not Hamiltonian. Give a reason why it does not have a Hamiltonian cycle.

(c) Generalize: find and prove a rule that tells you when $K_{a,b,c}$ is Hamiltonian. You may assume $a \leq b \leq c$.