Math 3322: Graph Theory[1] Mikhail Lavrov

Lecture 21: Planarity testing

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1 Triangulations

1.1 The number of edges in a planar graph

Last time, we proved Euler's formula: in a connected plane embedding with n vertices, m edges, and f faces, $n - m + f = 2$. We also know that if we sum the lengths of all faces, then we get $2m$: twice the number of edges. Now we will use these ideas to bound the number of edges in a planar graph.

For this, we will have to restrict our attention to simple graphs only, even though Euler's formula is true for multigraphs as well. There is just no hope of getting any results otherwise: adding loops or parallel edges lets us increase the number of edges as much as we like, but it will never interfere with planarity.

What distinguishes planar graphs from planar multigraphs? It is the following claim: in a plane embedding of a simple graph with at least 2 edges, every face has length at least 3. There are really two cases here:

- If the plane embedding has multiple faces, they must be separated somehow; the only way to do this is with a closed curve in the plane. If the boundary of a face contains a closed curve, this corresponds to a cycle in the graph—and in a simple graph, every cycle has length at least 3.
- If the plane embedding has only one face, it has length $2m$. When the graph has at least 2 edges, the length of this faces is also at least 3 (and we can even say that it's at least 4).

When this fact applies, if we sum the lengths of all f faces, we get *at least* $3f$. This gives us an inequality between f and m (assuming $n \geq 3$): $2m \geq 3f$. We can use this inequality to prove the following theorem:

Theorem 1.1. *If G is a planar graph with $n \geq 3$ vertices and m edges, then $m \leq 3n - 6$.*

Proof. We may assume that G is connected; if not, we can add some edges to a plane embedding of G to connect it without ruining planarity. Since $n \geq 3$ and G is connected, $m \geq n - 1 \geq 2$, so it is valid to apply our reasoning above: every face in a plane embedding of G has length at least 3.

Combining Euler's formula $n - m + f = 2$ with the inequality $2m \geq 3f$, we get

$$2 - n + m = f \leq \frac{2}{3}m$$

which we can rearrange to $\frac{1}{3}m \leq n - 2$, or $m \leq 3n - 6$. □

[1]This document comes from an archive of the Math 3322 course webpage: http://misha.fish/archive/3322-fall-2024

Theorem 1.1 lets us immediately conclude that some graphs are not planar. For example, the complete graph K_5 has $n = 5$ vertices and $m = 10$ edges. We have $10 > 3 \cdot 5 - 6$; therefore K_5 cannot have a plane embedding.

Note, however, that Theorem 1.1 is a **necessary** condition, not a **sufficient** one. If $m \le 3n - 6$, we cannot conclude that a graph is planar! Here's a boring example: take K_5, and then add 95 isolated vertices. Here, $n = 100$ and $m = 10$, so m is much less than $3n - 6$; but the graph is still not planar. (We will see less boring examples later today.)

1.2 Looking at the extreme cases

Whenever we prove an inequality, a natural question to ask is: what can we say about the cases where equality holds? What kind of planar graphs have $m = 3n - 6$?

To draw conclusions about such graphs, we should look back at our proof, and look at every place where an inequality appeared:

1. We said "We may assume G is connected" on the basis that if it's, not, we can get a connected planar graph with the same number of vertices, but more edges.

   So if a planar graph satisfies $m = 3n - 6$, it *must* be connected.

2. Our inequality $m \le 3n - 6$ came from the inequality $2m \ge 3f$. If we had $2m > 3f$, we'd have gotten $m < 3n - 6$, instead, by the same argument.

   So if a planar graph satisfies $m = 3n - 6$, it must satisfy $2m = 3f$.

3. Our argument for $2m \ge 3f$ came from an inequality we only stated in words: every face has length *at least* 3.

   So if a planar graph satisfies $m = 3n - 6$ (and therefore satisfies $2m = 3f$), then every face must have length *exactly* 3 (in every plane embedding).

Such a plane embedding (a connected plane embedding in which all faces are triangles) is called a **triangulation**. In fact, we can show that:

Corollary 1.2. *For a planar graph G with $n \ge 3$ vertices, the following are equivalent:*

(i) G has $3n - 6$ edges.

(ii) Every plane embedding of G is a triangulation.

(iii) G is a maximal planar graph: if we add any edge to G, it stops being planar.

Proof. We have already proven that (i) $\iff$ (ii). We have exactly $3n - 6$ edges if and only if every inequality in the proof of Theorem 1.1 is an equality: if and only if every face has length exactly 3.

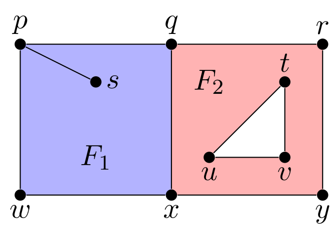
We also have (i) $\implies$ (iii) just from the inequality $m \le 3n - 6$. If a graph has exactly $3n - 6$ edges, and we add an edge, it has more than $3n - 6$ edges, so it can no longer be planar.

The new part is that (iii) also implies (i) and (ii): there are no maximal planar graphs that "get stuck" before becoming a triangulation with $3n - 6$ edges. We will show that (iii) implies (ii)...

... by showing the contrapositive: if G has an embedding that's *not* a triangulation, then G is *not* a maximal planar graph. Essentially, the argument is that if a plane embedding of G has a face F with $\mathrm{len}(F) \geq 4$, then we can add an edge between two of its vertices, drawing it inside F.

A few cases have to be considered to show that we can pick an edge that does not already exist *outside* F. This is tedious and not particularly educational, but I'm including it in these notes for completeness.

- If F is a cycle of length at least 5, then not all "chords" of that cycle can be edges of G. Otherwise, the vertices of the cycle would induce a K_n subgraph for $n \geq 5$. However, we already know that such graphs are not planar.
- If F is a cycle of length 4, and the two "chords" both existed outside the cycle, they'd have to cross. How do we know this? If they didn't cross, we could create a plane embedding of K_5 by putting a new vertex in the middle of F adjacent to all its vertices. But we know that K_5 is not a planar graph.
- If F is not a cycle, then it has an "outside" cycle and one or more vertices on the inside—as in the diagram I'm including below from an earlier lecture.



  Each vertex on the boundary of F that's strictly inside F is connected to the outside cycle by at most one edge, so we can draw any of the other edges to the outside cycle.

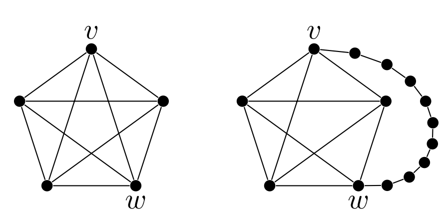
In all cases where $\mathrm{len}(F) \geq 4$, we have found an additional edge we can draw and still have a plane embedding. □

2 Subdivisions

2.1 Some more graphs that are not planar

Earlier today, we saw that K_5 was not planar, because it has too many edges.

We can get another example of a nonplanar graph as follows: take an edge vw of K_5, and replace it by a long $v - w$ path through entirely new vertices. The "before" and "after" of this procedure are shown below:



The second graph here has 9 more vertices and 9 more edges than K_5: $n = 14$ and $m = 19$. This comfortably satisfies the inequality $m \leq 3n - 6$.

However, the second graph is still not planar. Any plane embedding of the second graph would immediately give us a plane embedding of K_5: just replace the drawing of the long $v - w$ path by a drawing of the edge vw that traces out the same curve in the plane. Since K_5 is not planar, the second graph cannot be planar, either.

The general notion here is called a "subdivision". To **subdivide** an edge vw means to create a new vertex x, and replace edge vw by edges vx and xw. A **subdivision** of a graph G is a graph H obtained from G by subdividing edges some number of times. To make the statement of Kuratowski's theorem simpler later, we say that G itself is also a subdivision G.

(In the example above, we subdivide edge vw, then subdivide the new edges created; every time we subdivide an edge along the $v - w$ path, it makes the path longer.)

For the same reasons as with the first example, if H is a subdivision of G, then they are either both planar or both not planar.

2.2 Kuratowski's theorem

So far, we have shown two graphs to be nonplanar: K_5 and $K_{3,3}$. As a consequence, a subdivision of K_5 or $K_{3,3}$ cannot be planar. Moreover, if a graph G *contains* a subdivision of K_5 or $K_{3,3}$ as a subgraph, then G cannot be planar: we can't even find a plane embedding of that subgraph of G, much less all of G.

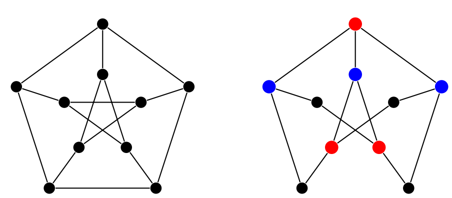
The reason I emphasize K_5 and $K_{3,3}$ in particular is because of the following theorem (which we will state, but not prove, because the proof is very long).

Theorem 2.1 (Kuratowski)**.** *If a graph G is not planar, then G contains a subdivision of K_5 or $K_{3,3}$ as a subgraph.*

This is a sort of "guarantee of proof" theorem. In principle, it is easy to give a proof that G is a planar graph: just draw a plane embedding of G. (*Finding* the plane embedding may, admittedly, be very hard. But at least we know that if G is planar, then such a demonstration exists.) However, proving that G is not a planar graph could be hard. Kuratowski's theorem says that if G is not planar, then we can always point out a subgraph of G that is a subdivision of $K_{3,3}$ or K_5, and then we have a proof that G is not planar. For example:

Claim 2.2. *The Petersen graph is not planar.*

Proof. Here is a subdivision of $K_{3,3}$ inside the Petersen graph:



By Kuratowski's theorem, the Petersen graph is not planar. □

Okay, but how do we find this subdivision? Some part of the process is creativity, but there are standard tricks we can try.

- Any subdivision of K_5 contains five vertices of degree 4: the vertices corresponding to the vertices of the original K_5. The Petersen graph does not have *any* such vertices, so it cannot contain a subdivision of K_5.

  This tells us that we must be looking for a subdivision of $K_{3,3}$, instead.

- In general, we'd want to pick the high-degree vertices to play the key roles in this subdivision. This doesn't help us here, though, because all vertices of the Petersen graph are identical.

- It can help to think of a subdivision of $K_{3,3}$ as picking vertices $v_1, v_2, v_3, w_1, w_2, w_3$, and then finding nine paths: a $v_i - w_j$ path for every i and j. These paths cannot share any of their vertices apart from the endpoints.

  To find *nine* such paths in a graph this small, most of the paths must be very short. So it makes sense, at least as a first try, to pick an arbitrary vertex to be v_1 and then its neighbors to be w_1, w_2, w_3. That takes care of the $v_1 - w_1$, $v_1 - w_2$, and $v_1 - w_3$ paths; we just have to locate v_2 and v_3.

- We can also think of $K_{3,3}$ and K_5 as cycles with some additional edges. Specifically, $K_{3,3}$ is a cycle $(v_1, w_1, v_2, w_2, v_3, w_3, v_1)$ with the extra edges v_1w_2, v_2w_3, v_3w_1. K_5 is a cycle $(v_1, v_2, v_3, v_4, v_5, v_1)$ with the extra edges $v_1v_3, v_1v_4, v_2v_4, v_2v_5, v_3v_5$.

  So we can start our process by finding a long-ish cycle: for example, we can start with a cycle of length 8 or 9 in the Petersen graph if we want to recover the subdivision in the diagram. Then, find paths to take the place of the "extra edges".

3 Extensions of the triangulation bound

The test for planarity that we get from Kuratowski's theorem is useful because it's guaranteed to give us an answer. However, the test of Theorem 1.1 is more convenient, when it applies, because it's much easier to count edges than to look for subdivisions.

One thing that we can do to make the test more powerful is to look for situations when the upper bound is stronger than $3n - 6$.

3.1 Bipartite planar graphs

Let G be a bipartite planar graph. Then what can we say about face lengths in a plane embedding of G?

They must be even: the boundary of a face consists of one or more closed walks, and in a bipartite graph, all closed walks have even length. In a connected, simple graph with at least 3 vertices, the smallest even length of a face is 4. This lets us replace the inequality $2m \geq 3f$ in the proof of Theorem 1.1 with a stronger inequality: $2m \geq 4f$.

We can, as before, combine this inequality with $n - m + f = 2$ (assuming G is connected, which we can always do in an upper bound). This gives us

$$2 - n + m = f \leq \frac{2}{4}m.$$

Simplifying, we conclude the following:

Theorem 3.1. *If G is a planar bipartite graph with $n \geq 3$ vertices and m edges, then $m \leq 2n-4$.*

To appreciate the strength of Theorem 3.1 as compared to Theorem 1.1, we can use it to prove a result from last time more easily. Theorem 3.1 implies that $K_{3,3}$ is not planar! $K_{3,3}$ has $n = 6$ vertices and $m = 9$ edges, which exceeds the upper bound of $2n - 4 = 8$.

3.2 Optional: girth and planarity

We can further generalize Theorem 3.1 by looking at shortest cycles. Let's define the **girth** of a graph G to be the length of the shortest cycle in G. (This is always at least 3, and in bipartite graphs it is always at least 4.) In an acyclic graph, the girth is sometimes defined to be ∞, but that will not serve our purposes today, and in any case, we don't need a planarity test for acyclic graphs: all forests are planar.

Theorem 3.2. *Let G be a planar graph with at least one cycle.*

If G has $n \geq 3$ vertices, m edges, and girth g, then $m \leq \frac{g}{g-2}(n-2)$.

Proof. First of all, let's carefully look at what the girth tells us about the length of a face. In the very nicest case, the boundary of each face is a cycle. Then, the girth tells us that every face in a plane embedding of G has girth at least g. However, things might not be so nice: the boundary might consist of multiple closed walks.

Fix a particular face F, and let G' be the subgraph of G which we obtain by deleting every edge which appears twice in the boundary of F. Then G' also has a plane embedding we can draw directly from the plane embedding of G, and F is also a face of that plane embedding: the edges we deleted separate F from F, not F from any other face. The boundary of F has only gotten shorter in G'. But (with an exception we'll get to later) in G', the boundary of F consists of one or more *cycles*, which all have length at least g; therefore F has length at least g as well, in G' and in the original graph G.

The exception is this. More precisely, we can say that in G', every remaining component of the boundary of F is a cycle. But what if F has no boundary left in G' at all? What if G' is just a collection of isolated vertices? This happens if we started out with a graph G that was a forest—in which F is the only face. For this reason, we add a condition to our theorem: G must have at least one cycle.

Now we can continue as before. Assume our graph G is connected, by adding edges to make it so, if necessary. Then we have two conditions: $n - m + f = 2$ from Euler's formula, and $2m \geq fg$, via the sum of face lengths. As before, we get an inequality

$$2 - n + fm = f \leq \frac{2}{g} \cdot m$$

which simplifies to $m \leq \frac{g}{g-2}(n-2)$, the inequality we wanted. ∎

4 Practice problems

1. The two graphs below were used as an example in the previous lecture.

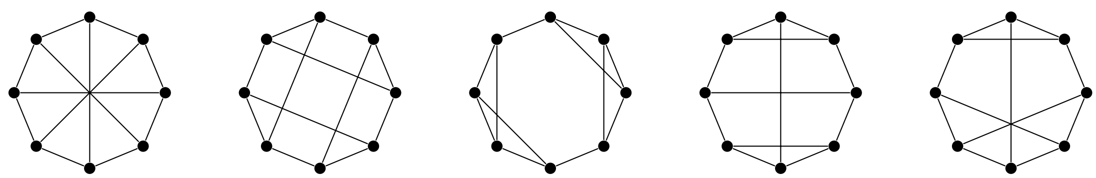
   One of these is planar, and the other one is not.

   (a) Identify the planar graph, and draw a plane embedding.

   (b) For the other graph, find a subdivision of $K_{3,3}$ or K_5 to show that it is not planar.

2. Take Theorem 3.2 for a spin by using it to prove that the Petersen graph is not planar.

3. Determine which of the five connected 3-regular graphs (all shown below) are planar, and which are not.



4. Find two *non-isomorphic* planar graphs with 6 vertices and $12 = 3 \cdot 6 - 6$ edges, and prove that they are not isomorphic.

5. What is the maximum number of edges in an n-vertex planar graph if we know it has a plane embedding with two faces of length 6?

6. Let G be a graph with n vertices and $n + 3$ edges obtained by starting with the cycle graph C_n and adding 3 more edges.

   When is G planar, and when is G not planar?

7. An *outerplanar graph* is one which has a plane embedding in which all the vertices lie on the outer face (the unbounded one).

   (a) Prove an upper bound on the number of edges in an n-vertex outerplanar graph. (You may assume $n \geq 2$; when $n = 1$ the upper bound is of course 0.)

   (b) Prove that K_4 and $K_{2,3}$ are not outerplanar graphs. ($K_{2,3}$ is trickier.)