Math 3322: Graph Theory[1] Mikhail Lavrov

Lecture 22: Polyhedra

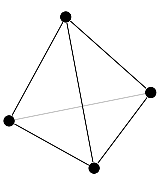
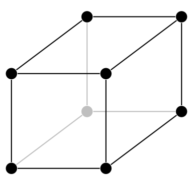
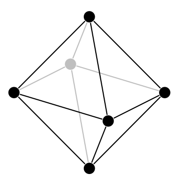
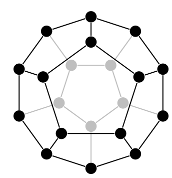
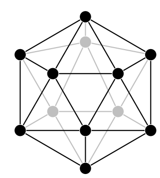
October 31, 2024 *Kennesaw State University*

1 Polyhedra

1.1 Polyhedra and graphs

A **polyhedron** (plural: polyhedra) is the 3-dimensional version of a polygon: it's a 3D shape with polygonal sides. The sides meet at edges, and the edges meet at corners which are also called vertices. This is not a coincidence: if we have a polyhedron, we can form its **skeleton graph** whose vertices are the corners of the polyhedron, and whose edges are the geometrical edges of the polyhedron.

We have already seen the cube graph Q_3 in many examples; this is the skeleton graph of a cube. As we will prove today, the cube is one of five Platonic solids: polyhedra whose faces are identical regular polygons, with the same number of faces meeting at each vertex. Here they are:

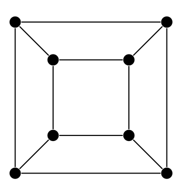
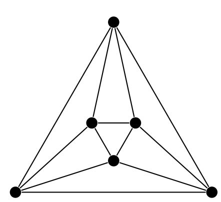






Tetrahedron Cube Octahedron Dodecahedron Icosahedron

What's the connection to things we've done before in this class? Well, all five of these graphs are planar graphs. (More precisely, whenever we have a polyhedron with no holes in it—one that can be drawn on the surface of a sphere—the skeleton graph is a planar graph.)

To get a plane embedding of one of these graphs, you should imagine taking one face of the polyhedron and stretching it out until everything else fits inside it. Alternatively, with care, you can draw the embedding directly, just by knowing how many sides the faces have, and how many faces meet at every vertex.

For example, here are the plane embeddings of the cube and octahedron, which are not too messy:

[1]This document comes from an archive of the Math 3322 course webpage: http://misha.fish/archive/3322-fall-2024

1.2 Classifying the Platonic solids

In two dimensions, there are infinitely many regular polygons. So why are there only five Platonic solids in three dimensions? This is, in part, something we can prove from Euler's formula. (However, read the note at the end of this section!)

We can describe a Platonic solid by a pair (p, q) where every face has p sides, and q faces meet at every vertex. Geometrically, we must have $p \geq 3$ and $q \geq 3$. Then we can narrow down the options for p and q:

Theorem 1.1. *There are only five possibilities for the pair (p, q) in a Platonic solid.*

Proof. We can write down two equations for n (the number of vertices), m (the number of edges), and f (the number of faces) in terms of p and q.

- The graph is a q-regular graph, so by the degree sum formula, $nq = 2m$.
- Every faces has length p, so by the face length sum formula, $fp = 2m$.

We also have Euler's formula: $n - m + f = 2$. Replacing n by $\frac{2m}{q}$ and f by $\frac{2m}{p}$, we get

$$\frac{2m}{q} - m + \frac{2m}{p} = 2 \implies \frac{1}{q} - \frac{1}{2} + \frac{1}{p} = \frac{1}{m}.$$

From here, the constraint that lets us narrow down the pairs (p, q) is that $\frac{1}{m} > 0$. Therefore $\frac{1}{q} - \frac{1}{2} + \frac{1}{p} > 0$, or $\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$.

How can we get a total bigger than $\frac{1}{2}$ here? Let's do casework on p:

- If $p = 3$ (every face is a triangle) then $\frac{1}{q} > \frac{1}{2} - \frac{1}{p} = \frac{1}{6}$, so $q < 6$.

  We can have $q = 3$ (three triangles meet at every vertex), giving us the tetrahedron.

  We can have $q = 4$ (four triangles meet at every vertex), giving us the octahedron.

  We can have $q = 5$ (five triangles meet at every vertex), giving us the icosahedron.

- If $p = 4$ (every face is a square) then $\frac{1}{q} > \frac{1}{2} - \frac{1}{p} = \frac{1}{4}$, so $q < 4$.

  We can have $q = 3$ (three squares meet at every vertex), giving us the cube.

- If $p = 5$ (every face is a pentagon) then $\frac{1}{q} > \frac{1}{2} - \frac{1}{p} = 0.3$, so $q < \frac{1}{0.3} = 3\frac{1}{3}$.

  We can have $q = 3$ (three pentagons meet at every vertex), giving us the dodecahedron.

These are the only possibilities: if $p \geq 6$, then even $q = 3$ does not satisfy $\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$. □

In each of these cases, we can use Euler's formula to solve for n, m, and f. For example, in the case of the dodecahedron, we know that $3n = 2m = 5f$. We could write everything in terms of m: $n = \frac{2}{3}m$, and $f = \frac{2}{5}m$. Putting this in Euler's formula, we get:

$$n - m + f = 2 \implies \frac{2}{3}m - m + \frac{2}{5}m = 2 \implies \frac{1}{15}m = 2 \implies m = 30.$$

So the dodecahedron has 30 edges. Since $n = \frac{2}{3}m$ and $f = \frac{2}{5}m$, it has 20 vertices and 12 sides.

Here is the complete table (where $\langle q \rangle \times n$ stands for the sequence $q, q, \ldots, q$ of length n):

	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Number of vertices	4	8	6	20	12
Degree sequence	$\langle 3 \rangle \times 4$	$\langle 3 \rangle \times 8$	$\langle 4 \rangle \times 6$	$\langle 5 \rangle \times 12$	$\langle 3 \rangle \times 20$
Number of edges	6	12	12	30	30
Number of faces	4	6	8	12	20
Face types	$\triangle \times 4$	$\square \times 6$	$\triangle \times 8$	$\pentagon \times 12$	$\triangle \times 20$

You may notice that the numbers in the "cube" and "octahedron" columns are the same, just rearranged; the "dodecahedron" and "icosahedron" columns are related similarly. This is not a coincidence! We will explain it in the next section.

One final note: though Theorem 1.1 guarantees that (p, q) is one of $\{(3, 3), (3, 4), (3, 5), (4, 4), (5, 3)\}$, and we can solve for n, m, f in terms of (p, q), it is theoretically posssible that there are multiple planar graphs with the same parameters n, m, f, p, q. Could there be a second icosahedron where the faces attach differently?

It turns out there is only one possibility for each (p, q). See the practice problems for ideas about how to prove this for the tetrahedron, cube, and octahedron. For the icosahedron and dodecahedron, the argument is more complicated, though it boils down to checking finitely many cases.

2 Dual graphs

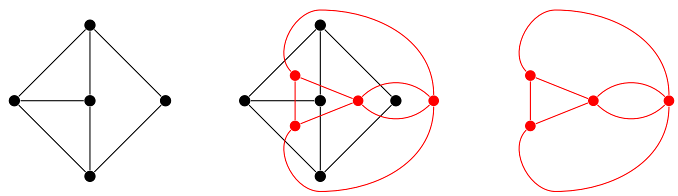
2.1 Definition

You may have noticed that the face length formula for plane embeddings is very similar to the degree sum formula for graphs:

$$\sum_{i=1}^{f} \text{len}(F_i) = 2m \qquad \sum_{i=1}^{n} \deg(v_i) = 2m.$$

This is also not a coincidence! The face length formula is just the degree sum formula applied to a graph called the **dual graph** of the plane embedding, whose vertices are the faces $F_1, F_2, \ldots$. We define the dual graph to have an edge F_iF_j whenever faces F_i and F_j touch along an edge in the plane embedding we started with.

Here is an example, with the original plane embedding in black and the dual graph in red. If we're careful, we can use the plane embedding of the original graph to find a plane embedding of the dual graph. Just place the dual vertex corresponding to face F_i somewhere in the interior of face F_i, and have the dual edges cross the edges of the original embedding.



However, you also notice from this picture that the dual graph is... not, strictly speaking a graph. It is a multigraph: when two faces touch along multiple edges, the dual graph has multiple edges between the corresponding vertices. Sometimes the dual graph has loops, when an edge has the same face on both sides.

It is also important to remember that saying "G^* is the dual graph of G" is not, strictly speaking, correct. The dual graph is defined in terms of a plane embedding of G; if we choose a different plane embedding, we may get a different dual graph.

Some things do stay the same however. If G is a planar graph, then we know that the number of vertices, edges, and faces is constant over all plane embeddings of G. In G^*, the number of edges is the same; however, the number of vertices in G^* is the number of faces in G, and the number of faces in G^* is the number of vertices in G.

2.2 Dual polyhedra

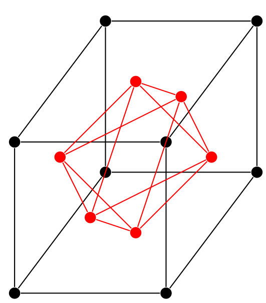
Suppose our planar graph G is the skeleton graph of a polyhedron. In this case, we can construct G^* in a way that reflects the polyhedron's geometry. Let's do the following:

1. For the dual vertex corresponding to face F_i of the polyhedron, draw a point in the center of face F_i.
2. For the dual edge connecting adjacent faces F_i and F_j, draw a line segment between the two points in the centers of F_i and F_j.

This is still the same procedure we used above for finding a dual graph, we're just being more specific about what these vertices and edges look like.

If the polyhedron is sufficiently symmetric—for example, in the case of all five Platonic solids—this dua construction is another polyhedron. (If the polyhedron is not symmetric enough, then its faces might not "lie flat" when we take this dual. There are other, more complicated geometric constructions that make sense for a larger variety of polyhedra. For all of them, the underlying graph-theoretic idea is dual of a plane embedding defined earlier.)

For example, if we draw a dual of the cube, we get the octahedron:



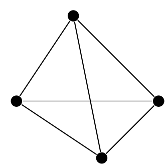
This explains why the parameters of the cube and octahedron are very similar: the skeleton graphs are dual graphs! Similarly, the skeleton graphs of the dodecahedron and icosahedron are duals. The tetrahedron's skeleton graph, K_4, is its own dual.

3 More general polyhedra (optional)

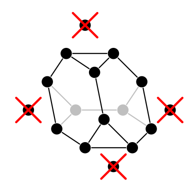
The definition of a Platonic solid is the most restrictive generalization of a regular polygon to three dimensions. A slightly less restrictive, and still very interesting, definition is that of an **Archimedean solid**.

These are convex polyhedra whose faces are all regular polygons, and whose vertices are all symmetric to each other (that is, for any two vertices, there is some rotation or reflection of the polyhedron that can move one to the other). Notably missing from this definition is any kind of symmetry between faces: in an Archimedean solid, the faces do not all have to be the same!

Here is an example, the **truncated tetrahedron**. Geometrically, this is obtained as follows: start with a tetrahedron, and cut off each vertex a third of the way along its edge, as shown in the picture below.



Tetrahedron



Truncated Tetrahedron

The truncated tetrahedron has two types of faces: four hexagons (left over from the original faces of the tetrahedron) and four triangles (from where the cuts were made).

As with the Platonic solids, we can *at the very least* determine the global face, vertex, and edge counts from a local description of what is happening at every vertex. Let's see how, using the truncated tetrahedron as an example. (Imagine that we don't have the picture above to use as a reference.)

Suppose that we know that at every vertex of the truncated tetrahedron, two hexagons and a triangle meet. We can set up equations in the following variables:

1. m, the number of edges.
2. n, the number of vertices.
3. f_3, the number of 3-sided faces (triangles).
4. f_6, the number of 6-sided faces (hexagons).

Here's how we can do this. As before, we have Euler's formula, telling us that $n-m+(f_3+f_6)=2$. Since three faces total meet at each vertex, we know that every vertex has degree 3 in the skeleton graph, so $2m=3n$. The new trick is how we count the faces in terms of the vertices:

- Each of the n vertices is the corner of two hexagons, so if we go through each vertex and list the hexagons that meet there, we will make a list of $2n$ hexagons. However, each hexagon will appear on this list 6 times: once for each of its corners. Therefore $2n = 6f_6$.
- Similarly, each of the n vertices is the corner of one triangle, so if we try to list the triangles by going through each vertex and writing down the triangle that has a corner there, our list

will have n triangles on it. However, each triangle will appear on this list 3 times: once for each of its corners. Therefore $n = 3f_3$.

(In general, these equations are determined by two quantities: the number of sides each type of face has, and the number of faces of that type that meet at each vertex.)

Now we can write Euler's formula solely in terms of n, by replacing each variable by a multiple of n:

$$n - m + f_3 + f_6 = 2 \implies n - \frac{3}{2}n + \frac{1}{3}n + \frac{1}{3}n = 2.$$

This simplifies to $\frac{1}{6}n = 2$, or $n = 12$. From here, we can determine that $m = 18$, $f_3 = 4$, and $f_6 = 4$: exactly the parameters of a truncated tetrahedron!

There is a way to make this process more systematic—and also generalize it to be able to deal with even less regular polyhedra.

To do so, we define the **angle defect** at a corner of a polyhedron to be 2π minus the sum of the angles of the polygons meeting at that corner. Since the angle defect would always be 0 if the corner were flat, this is a measure of how much the polyhedron "bends" at a corner. It's a very nice measure, due to the following theorem:

Theorem 3.1 (Descartes's formula)**.** *In any convex polyhedron, the sum of all angle defects is* 4π*.*

Proof. This could be done by solving a system of equations, but there is a more elegant proof by something called the "discharging method". We take the skeleton graph of the polyhedron, and put a "charge" of $+2\pi$ on each vertex, $+2\pi$ on each face, and -2π on each edge. By Euler's formula, the total charge on the graph is $2\pi n - 2\pi m + 2\pi f = 4\pi$.

Just like positive and negative electric charges cancel, we will move around these charges to cancel them, while not changing the overall sum. First, from each face, we move $+\pi$ charge on to each of its edges. This leaves each edge at charge 0; it started at -2π, but gained $+\pi$ from each of the two faces it borders. However, each face has now gone into the negatives: a face of length ℓ now has charge $-(\ell - 2)\pi$.

Recall from high school geometry that $(\ell - 2)\pi$ is the sum of the angles of an ℓ-sided polygon.[2] So we can bring each face up to neutral with the following second transformation: for every corner of every face, if that corner makes an angle of θ on that face in the polyhedron, we move θ charge from the vertex at that corner to the face.

When we're done, the faces and edges all have charge 0, while the remaining charge at each vertex is exactly the angle defect. However, the sum of the charges has remained at 4π throughout, proving the formula. □

Theorem 3.1 can quickly count the vertices in any Archimedean solid. For example, in the truncated tetrahedron, a regular triangle (with angle measure $\pi/3$) and two regular hexagons (with angle measure $2\pi/3$) meet at each vertex, so the angle defect at each vertex is $2\pi - \pi/3 - 2(2\pi/3) = \pi/3$. The total angle defect is 4π, so there must be $\frac{4\pi}{\pi/3} = 12$ vertices.

[2] To help you recall it, here is the sketch of an argument for this formula: we can always draw in some diagonals to separate an ℓ-sided polygon into $\ell - 2$ triangles, each with a sum of angles of π.

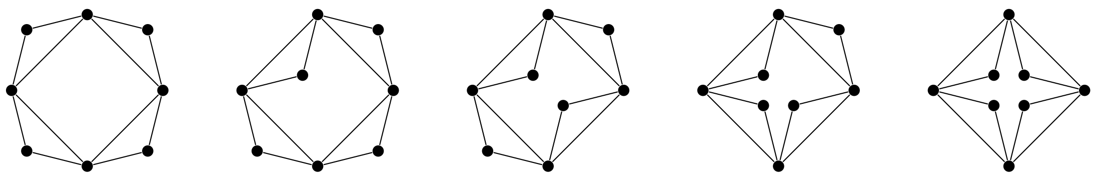
4 Practice problems

1. Draw a plane embedding of the skeleton graph of the dodecahedron.

2. The skeleton graph of the icosahedron is pancyclic: it has a cycle of every length from 3 to 12. Verify this by finding each of those cycles.

3. Determine what the dual graph of any embedding of any n-vertex tree looks like. (Conclude that two planar graphs with isomorphic duals are not, themselves, necessarily isomorphic.)

4. Here are five different plane embeddings of a graph G, from two lectures ago:

   For each plane embedding, draw the dual graph. Determine which of these graphs are isomorphic to each other, and which are not.

   Can you think of a way in which the plane embeddings with isomorphic dual graphs are somehow "the same", even though they do not look the same?

5. A **uniform n-gonal prism** is a prism whose top and bottom are regular n-gons, and whose sides are n squares.

   (a) Draw a plane embedding of the skeleton graph of a uniform n-gonal prism where n is some very big number—like 6. Explain how to draw such a plane embedding for any value of n.

   (b) Draw the dual graph of the plane embedding you drew in the first part. Describe the structure of this dual graph for arbitrary values of n.

   (c) Geometrically, what does the dual polyhedron of the n-gonal prism look like?

6. Here are few more questions about Archimedean solids.

   (a) An **icosidodecahedron** is an Archimedean solid with 12 pentagonal faces (like a dodecahedron) *and* 20 triangular faces (like an icosahedron). How many vertices and edges does it have? How many faces of each type meet at each vertex?

   (b) A **snub cube** is an Archimedean solid with four triangles and one square meeting at every vertex. How many vertices and edges does it have, and how many faces of each type?

   (c) What about the **truncated icosidodecahedron**, in which a 4-sided face, a 6-sided face, and a 10-sided face meet at every vertex?

   (d) The prisms mentioned in the previous question fall under the definition of an Archimedean solid, but are often excluded because there's infinitely many of them.

   There is another infinite family: the **uniform n-gonal antiprisms**.

These have two faces that are regular n-gons, and all other faces are triangles. At each vertex, one of the n-gons and k of the triangles meet. What is the value of k? How many triangular faces are there in total?

7. Suppose that G is an n-vertex planar multigraph such that (for at least one plane embedding of G) the dual graph G^* is isomorphic to G. (We call such a multigraph **self-dual**.)
   (a) How many edges must G have, in terms of n?
   (b) Find an example of such a graph G for all $n \geq 2$.

8. (a) It follows from Theorem 1.1 that any Platonic solid with $(p, q) = (3, 3)$ is a 3-regular 4-vertex graph. Prove that there is only one such graph (up to isomorphism), and conclude that the tetrahedral graph is unique.
   (b) It follows from Theorem 1.1 that any Platonic solid with $(p, q) = (3, 4)$ is a 4-regular 6-vertex graph. Prove that there is only one such graph (up to isomorphism), and conclude that the tetrahedral graph is unique.

9. For the cube graph, we need to know a bit more.
   (a) Suppose that G is a planar graph with a plane embedding in which every face has even length. Prove that G is bipartite.

   (Hint: for any arbitrary cycle C, get a new plane embedding of some subgraph H by finding C inside the plane embedding of G, and erasing everything outside that cycle. What can you say about the lengths of the faces in this embedding, and why does this prove that C has even length?)

   (b) It follows from Theorem 1.1 that any Platonic solid with $(p, q) = (4, 3)$ is a 3-regular 8-vertex graph; from part (a), it follows that the graph is bipartite. Prove that there is only one such graph (up to isomorphism), and conclude that the cube graph is unique.