Math 3322: Graph Theory[1] Mikhail Lavrov

Lecture 26: Cut vertices

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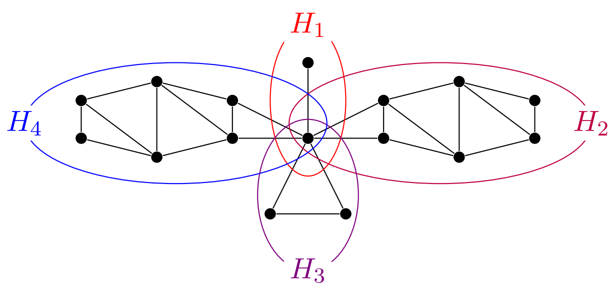
1 Cut vertices

Near the beginning of the semester, we discussed connected components of graphs. For many graph-theoretical problems we've solved since then, these components don't interact at all: we can solve the problem for a graph with many connected components by solving it on each component separately.

One step up from this, but very similar in flavor, are graphs with cut vertices. Formally:

- If G is a connected graph, a vertex $v \in V(G)$ is a **cut vertex** if $G - v$ is not connected.
- Very occasionally, we might want to talk about cut vertices when G is not connected. In that case, it makes sense to say that v is a cut vertex if $G - v$ has more connected components than G.

Let's see how this helps us break a problem down into simpler problems. In all of these, let's say that $H_1, H_2, \ldots, H_k$ are the connected components of $G - v$, *but with* v *added back in.* Here's a picture to help make this make more sense:



All the graphs $H_1, H_2, \ldots, H_k$ contain v, and they overlap only at v.

How does this help us break down a problem? Here are some examples.

- If our goal is to determine whether G is planar, it's enough to know whether $H_1, H_2, \ldots, H_k$ are all planar. If they are, then they each have an embedding where v is one of the vertices on the unbounded face. Then we can join the embeddings together at v like the petals of a flower (or like the graphs in the diagram above).
- If we want to find a proper coloring of G, it's enough to color $H_1, H_2, \ldots, H_k$. Relabel the colors in their colorings so that v has the same color in each of them. Then, we can use those colorings to color all of G.

  In particular, $\chi(G) = \max\{\chi(H_1), \chi(H_2), \ldots, \chi(H_k)\}$.

[1]This document comes from an archive of the Math 3322 course webpage: http://misha.fish/archive/3322-fall-2024

- If we want to find the largest clique in G, it's enough to find the largest clique in each of $H_1, H_2, \ldots, H_k$: a clique in G must be entirely contained in one of these.

  Just as with chromatic number, we have $\omega(G) = \max\{\omega(H_1), \omega(H_2), \ldots, \omega(H_k)\}$.

- Every spanning tree of G can be obtained by putting together a spanning tree of each of $H_1, H_2, \ldots, H_k$. This can be useful for finding a minimum-cost spanning tree, or by counting the number of possible spanning trees.

There are other problems in graph theory that become easier when we located a cut vertex; some we have talked about, and many we have not!

2 2-connected graphs

A graph G on at least 3 vertices is **2-connected** if G is connected and has no cut vertices: for all $v \in V(G)$, the graph $G - v$ is also connected.

(Why "at least 3 vertices"? This is added because the graph K_2 (with only 2 vertices) also has no cut vertices; we don't consider it 2-connected, since it lacks many of the properties of other 2-connected graphs.)

Knowing that a graph *does* have a cut vertex gives us a way to break it down into simpler pieces. But we might also want to understand 2-connected graphs, for two reasons:

- In practical problems, because they represent networks with some amount of resilience. For example, a 2-connected computer network is one that continues to be connected if something happens to one of the computers. A 2-connected airline network is one that continues to be connected if something happens to one of the airports.
- In theoretical problems, it lets us make extra use of cut vertices. If cut vertices let us simplify a particular problem, then it's enough to solve the problem for 2-connected graphs. Is this easier? That depends on how many properties of 2-connected graphs we know!

There are many properties of 2-connected graphs, but today we will see a fundamental one:

Theorem 2.1. *G is 2-connected if and only if any two vertices of G lie on a common cycle.*

The easy direction of this theorem is to show that if G has this property, it is 2-connected.

Suppose any two vertices of G lie on a common cycle, and we delete vertex v. Let u, w be any other vertices of G. Then since u and w lie on a common cycle in G, one of two things can happen in $G - v$:

- The cycle survives intact.
- The cycle contained v, and falls apart into a path containing u and w.

In either case, the cycle or whatever is left of it contains a $u - w$ path, so in particular there is a $u - w$ path in $G - v$. Since u and w were arbitrary, $G - v$ is connected; since v was arbitrary, G is 2-connected.

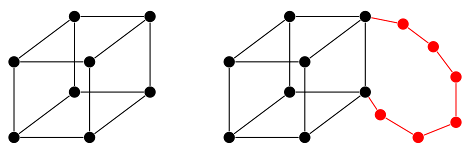
We will not prove the hard direction of Theorem 2.1 yet. Instead, we will pivot in a different direction, and return to the proof of this theorem at the end of the lecture.

3 Ear decompositions

Theorem 2.1 is a useful characterization of 2-connected graphs, but it's hard to use it to check if a graph is 2-connected. We'd have to find lots of cycles to demonstrate that there's one through any two vertices. That's not much easier than checking that $G-v$ is connected for every vertex v.

We would like to have a "short certificate" that a 2-connected graph really is 2-connected. One way to do this is through ear decompositions.

An **ear** of a graph G is a path[2] in G in which every vertex except the first and the last (every **internal** vertex) has degree 2. When we **add an ear** to a graph G, we pick two vertices v, w of G and create an ear by adding entirely new edges and (if the ear has length 2 or more) entirely new internal vertices to form a $v-w$ path. This is actually easier to show than to explain. Here is a cube graph, and a cube graph with an ear added:



In particular, adding an edge to G (and no new vertices) is adding an ear of length 1.

Lemma 3.1. *If G is a 2-connected graph and we add an ear to G, the resulting graph is also 2-connected.*

Proof. Let H be a graph obtained from G by adding a $v-w$ ear whose internal vertices are $x_1, x_2, \dots, x_k$. (It's possible that $k = 0$.) We will check that the new graph H still does not have a cut vertex. To do this, we see what happens when we delete a vertex of H:

- Suppose we delete a vertex $u \in V(G)$ other than v or w. Because $G-u$ is connected, all vertices of $G-u$ are in the same connected component of $H-u$. Also, $x_1, x_2, \dots, x_k$ all have a path to v and to w, so they are also in that same connected component: $H-u$ is connected.
- If we delete v or w, essentially the same thing happens. The only change is that for $x_1, x_2, \dots, x_k$, we should observe that we still have a path to whichever of v or w we didn't delete.
- If we delete one of the new vertices x_i, then G remains connected (we didn't touch it); vertices $x_1, \dots, x_{i-1}$ still have a path to v; vertices $x_{i+1}, \dots, x_k$ still have a path to w. As a result, $H - x_i$ is still connected.

In all cases, H has no cut vertices, so H is 2-connected. □

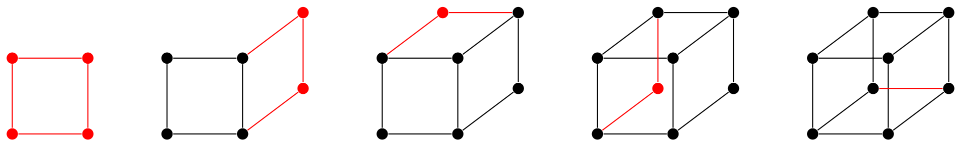
We can use Lemma 3.1 to prove that a graph is 2-connected. Suppose we start from a cycle graph: that's 2-connected, because deleting any vertex leaves a path. Then, we add an ear to this cycle graph (getting another 2-connected graph). Then, we repeatedly add ears. By Lemma 3.1, the result will always be 2-connected.

[2]For the purposes of this topic, we will switch back and forth between thinking of paths as sequences of vertices (as we defined them at the beginning of the semester) and as subgraphs of G.

An **ear decomposition** of G is a sequence of ear-adding steps that starts at a cycle graph and ends at G. This is a proof that G is 2-connected. Formally, an ear decomposition is a decomposition of G into a union $G = R_1 \cup R_2 \cup \cdots \cup R_k$[3] where:

- R_1 is a cycle.
- For each $i \geq 1$, R_{i+1} is an ear of $R_1 \cup R_2 \cup \cdots \cup R_i$.
- To be clear (this follows both from the definition of an ear, and from the definition of a decomposition) there are no edges shared between $R_1, R_2, \ldots, R_k$: each edge of G is in exactly one of these graphs.

For example, here is a proof by ear decomposition that the cube graph is 2-connected:



Does such a proof always exist? Yes!

Theorem 3.2. *If G is a 2-connected graph, then it has an ear decomposition.*

Proof. To prove this theorem, we have to reason in the opposite way from Lemma 3.1.

To find the ear decomposition $G = R_1 \cup R_2 \cup \cdots \cup R_k$, we can start by letting R_1 be any cycle in G. Why does a cycle exist? Well, we know G is connected and has at least three vertices. If G had no cycles, it would be a tree—but then, any non-leaf vertex would be a cut vertex, contradicting our assumption that G is 2-connected.

Next, suppose we've constructed $R_1 \cup R_2 \cup \cdots \cup R_i$, but it's still a proper subgraph of G. We'd like to be able to make it bigger, by adding an ear—but adding an ear that's still entirely contained in G.

Let V_i be the set of vertices included in $R_1, R_2, \ldots, R_i$. The first question is: is $V_i = V(G)$? If it is, then we're nearly done, and continuing the ear decomposition is easy. Pick any edge of G not in $R_1 \cup R_2 \cup \cdots \cup R_i$, and make that edge its own ear. (We will continue by adding the rest of the edges, one at a time.)

So suppose instead that $V_i \neq V(G)$. Because G is connected, it must have some edge xy where $x \in V_i$ and $y \notin V_i$. We will try to construct an ear that begins with the edge xy.

Because $G - x$ is also connected (that's what it means for G to be 2-connected!) we can find a path in $G - x$ from y to any other vertex we choose. Let's make that other vertex a vertex $z \in V_i$ (other than x). The graph $G - x$ must contain a $y - z$ path; how does this path help us?

Well, let $(v_0, v_1, v_2, \ldots, v_k)$ be the sequence of vertices along this path, with $v_0 = y$ and $v_k = z$. Since $v_0 \notin V_i$ and $v_k \in V_i$, there must be some positive integer j such that v_j is the *first* vertex of the path in V_i. (We might have $v_j = v_k = z$, or we might have $j < k$.)

[3] The R stands for "eaR". Silly, I know, but the letter E is usually reserved for edge sets.

Now let R_{i+1} be the path $(x, v_0, v_1, \ldots, v_j)$. By construction, this path starts and ends in V_i, but all its intermediate vertices are outside V_i, so it is in fact an ear of $R_1 \cup R_2 \cup \cdots \cup R_i$, as desired.

In both cases, we've constructed a new ear R_{i+1}, making $R_1 \cup R_2 \cup \cdots \cup R_{i+1}$ bigger. Keep going until we end up building all of G. $\square$

An important observation is that no matter how we've constructed $R_1 \cup R_2 \cup \cdots \cup R_i$, if we haven't finished, then we can always choose R_{i+1} somehow. This is what makes the algorithm in the proof a greedy one: at each step, it is enough to add *some* ear, and we don't have to worry about our earlier choices leading us to a dead end. (In particular, we do not have to follow the exact algorithm suggested in the proof for finding an ear, and for small graphs, doing so would just make our life needlessly complicated.)

4 From ears to Theorem 2.1

Having an ear decomposition of a graph is useful for practical purposes: it is a quick proof that our graph is 2-connected. It is also useful for theoretical purposes: many proofs for 2-connected graphs become simpler if we start with an ear decomposition. We'll see an example of this, by using the idea of ear decompositions to prove Theorem 2.1.

This will be a complicated proof, so we'll begin with some preliminary lemmas. The first is a boring one that we'll never need again, because Theorem 2.1 will replace it as soon as we've proved it.

Lemma 4.1. *If G is 2-connected, and v is any vertex, then G has a cycle containing v.*

Proof. If v only had one neighbor, then we could delete that neighbor and disconnect v from the rest of G, violating the assumption that G is 2-connected. Therefore v has at least 2 neighbors.

Let u and w be two neighbors of G. Because G is 2-connected, there is a $u - w$ path in $G - v$. Let $(v_0, v_1, v_2, \ldots, v_k)$ with $u = v_0$ and $w = v_k$ be that path. Then $(v, v_0, v_1, v_2, \ldots, v_k, v)$ is a cycle in G containing v. $\square$

The importance of Lemma 4.1 is that it lets us pick a particular ear decomposition: one in which vertex v is in the initial ear G_1.

The next lemma is not boring. It comes up often in the study of 2-connected graphs.

Lemma 4.2. *If G is 2-connected, and u, v, w are any three vertices, then G has a $u - w$ path that passes through v.*

Proof. By Lemma 4.1, there is a cycle containing v; call that cycle R_1. By Theorem 3.2, this cycle can be the first ear of an ear decomposition of G. What we'll do is prove by induction that as we build up G in such an ear decomposition, this lemma will hold for any two vertices u and w in the graph we've built.

Initially, that's the case. If u and w are any two vertices of the cycle R_1, then the cycle contains two $u - w$ paths, and one of those $u - w$ paths passes through v.

Suppose that the lemma holds for the graph $R_1 \cup R_2 \cup \cdots \cup R_i$, and we're adding the ear R_{i+1}. We consider the following cases.

Case 1a. Vertex u is a vertex of $R_1 \cup R_2 \cup \cdots \cup R_i$, but vertex w is an internal vertex of the new ear R_{i+1}.

In this case, let w' be any endpoint of R_{i+1}. Since w' already existed in $R_1 \cup R_2 \cup \cdots \cup R_i$, by our inductive hypothesis, there is a $u - w'$ path passing through v. By tacking on part of R_{i+1} at the end, we can extend this to a $u - w$ path passing through v.

Case 1b. Vertex u is an internal vertex of the new ear R_{i+1}, but vertex w is a vertex of $R_1 \cup R_2 \cup \cdots \cup R_i$. This case is identical to Case 1, with the names u and w swapped.

Case 2. Both u and w are internal vertices of the new ear R_{i+1}.

In this case, let u' and w' be the endpoints of G_{i+1}, chosen so that u' is closer to u than to w on the path, and w' is closer to w than to u. This means that inside R_{i+1}, there is a $u - u'$ path and a $w' - w$ path that do not intersect.

By our inductive hypothesis, there is a $u' - w'$ path in $R_1 \cup R_2 \cup \cdots \cup R_i$ that contains v. If we tack on the $u - u'$ path at the beginning and the $w' - w$ path at the end, we get a $u - w$ path that contains v.

In all cases, the lemma continues to be true when we add an ear; therefore it is true in G. $\square$

Now we are ready to prove the main theorem.

Proof of Theorem 2.1. Let x and y be two vertices of G; we want to show that there is a cycle in G containing both x and y.

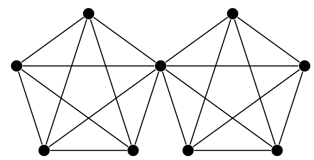
By Lemma 4.1, there is a cycle in G containing x; call that cycle R_1. If y also lies on R_1, we are done. Otherwise, by Theorem 3.2, R_1 can be the first ear of an ear decomposition of G; let's wait until y appears later in that ear decomposition, and see what happens.

Suppose that y appears when we're adding ear R_{i+1} to the graph $R_1 \cup R_2 \cup \cdots \cup R_i$. Let u and w be the endpoints of ear R_{i+1}. By Lemma 4.2, there is a $u - w$ path in $R_1 \cup R_2 \cup \cdots \cup R_i$ passing through x.

Combine this with R_{i+1} (another $u - w$ path, which has no vertices in common with the first) and we get a cycle. It still passes through x, and it also passes through y (because y lies on R_{i+1}), so it is the cycle we wanted. $\square$

5 Practice problems

1. Let G be a graph consisting of two copies of K_5 joined at a vertex:



How many spanning trees does G have? (Recall that K_n has n^{n-2} spanning trees.)

2. Suppose G has a cut vertex v, and that the graphs $H_1, H_2, \dots, H_k$ are defined as in the first section of these lecture notes.

   (a) Explain why knowing the independence numbers $\alpha(H_1), \alpha(H_2), \dots, \alpha(H_k)$ is not enough to find $\alpha(G)$.

   (b) Without knowing G, if all I tell you is the values of $\alpha(H_1), \alpha(H_2), \dots, \alpha(H_k)$, what are the minimum and maximum possible values of $\alpha(G)$?

3. Find an ear decomposition of:

   (a) $K_{3,3}$.

   (b) $K_{2,5}$.

   (c) $K_{2,n}$ for every n.

4. You might wonder: is there a shortest ear decomposition in a graph? In fact, every ear decomposition contains the same number of pieces.

   (a) Suppose that G has the ear decomposition $R_1 \cup R_2 \cup \dots \cup R_k$, where R_1 is a cycle of length ℓ_1 and for each $i \geq 2$, R_i is a path of length ℓ_i.

   Find the number of edges in G in terms of $\ell_1, \ell_2, \dots, \ell_k$. *(This is the easier part.)*

   (b) Find the number of vertices in G in terms of $\ell_1, \ell_2, \dots, \ell_k$. *(This is a bit trickier; think about how many vertices in each R_i are new to G.)*

   (c) If G has n vertices, m edges, and k ears in an ear decomposition, find a relationship between n, m, and k from your answers to the first two parts.

   Conclude that the value of k is predetermined by G, and doesn't depend on the ear decomposition.

5. One way to interpret Theorem 2.1 is that for any two vertices x, y in a 2-connected graph G, there are two $x - y$ paths that share none of their internal vertices. Here is an **incorrect** proof of a **false** generalization:

   Claim. If G is 2-connected and P is any $x - y$ path, there is a second $x - y$ path P' that shares no vertices with P other than x and y.

   "Proof". Suppose there is no such path P'. That means that all other paths P' end up sharing some other vertex of P. But then, deleting that vertex of P destroys all $x - y$

paths, and so that vertex was a cut vertex. This cannot happen in a 2-connected graph; contradiction! So a path P' must exist.

(a) Point out the mistake in the proof.

(b) Give a counterexample to the claim.

6. Prove that if G is 2-connected and e, e' are any two edges, then G has a cycle containing both e and e'.

7. The ear decompositions in this lecture are sometimes called **proper** ear decompositions. There is also a corresponding notion of **improper** ear decompositions. In the improper case, an ear of a graph G is also allowed[4] to be a cycle in which every vertex except one has degree 2. (That is, when we add an ear, we are allowed to start and end at the same vertex.)

   (a) Prove by example that a graph with an improper ear decomposition is not necessarily 2-connected.

   Instead, improper ear decompositions correspond to **2-edge-connected** graphs: graphs which remain connected when any edge is deleted. In other words, 2-edge-connected graphs are graphs with no bridges.

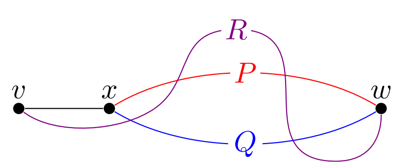
   (b) Prove that if a graph G has an improper ear decomposition, then it is 2-edge-connected. *(Recall that an edge is a bridge if and only if it does not lie on any cycles.)*

   (c) Prove that if a graph is 2-edge-connected, then it has an improper ear decomposition. *(You can follow the proof of Theorem 3.2, with some changes.)*

8. There is an alternate proof of Theorem 2.1 which does not use ear decompositions. Instead, we prove that any two vertices v, w lie on a common cycle by inducting on the distance $d(v, w)$. (I will phrase this in the equivalent way that there are two internally disjoint $v - w$ paths.)

   (a) For the base case, prove (without relying on any of our other results) that if G is two-connected, then for any edge $vw \in E(G)$, there is a cycle containing vw.

   (b) For the induction step, we assume that Theorem 2.1 holds for any two vertices at some distance $k \geq 1$, and let v, w be two vertices with $d(v, w) = k + 1$. To apply the inductive hypothesis, we let x be the first vertex on a shortest $v - w$ path, so that $d(x, w) = k$. Let P and Q be two internally disjoint $x - w$ paths, and let R be a $v - w$ path in $G - x$, as shown below:



   Prove that no matter how R intersects P and Q, we can find two internally-disjoint $v - w$ paths.

[4] An "improper" ear decomposition should maybe be called a "not-necessarily-proper" ear decomposition: we do not require it not to be proper.