Math 3322: Graph Theory[1] Mikhail Lavrov

Lecture 27: Connectivity

November 19, 2024 *Kennesaw State University*

1 Vertex cuts

In the previous lecture, we defined **cut vertices** and 2**-connected graphs**. Now we will generalize these definitions.

A **vertex cut** in a graph G is a subset $U \subseteq V(G)$ such that $G - U$ (that's the subgraph of G with the vertices in U, and all their incident edges, deleted) is no longer connected. If G is not connected, then the empty set is a vertex cut.

We would like to measure the connectivity of a graph by the number of vertices in the smallest vertex cut. There's a bit of a sticking point here: complete graphs don't have any vertex cuts! There is no way to turn K_n into a disconnected graph by deleting vertices.

So we have a bit of a funny definition. The **(vertex) connectivity** of a graph G, denoted $\kappa(G)$ is defined to be:

- The size of the smallest vertex cut of G, if one exists: if G is not a complete graph.
- $\kappa(K_n)$ is "artificially" set to $n-1$. We will see later why $n-1$ is the "right" value to choose.

In particular, $\kappa(G) = 0$ if G is not connected.

We say that a graph is k**-connected** if $\kappa(G)$ is *at least* k. This definition exists because we often want to say "Such-and-such result applies if our graph is connected *enough* for it to work".

A 2-connected graph is one with $\kappa(G) \geq 2$: a graph that has no vertex cut U with $|U| = 1$. Saying "$\{u\}$ is a vertex cut" is another way to say "u is a cut vertex", so a 2-connected graph is one that has no cut vertices. In other words, our new definition agrees with our old one.

1.1 Some basic properties of connectivity

In the previous lecture, we saw that the cube graph Q_3 is 2-connected. In fact, we can go one step further.

Claim 1.1. $\kappa(Q_3) = 3$.

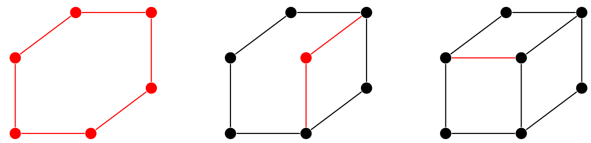
Proof. Ear decompositions were a nice way to prove that graphs are 2-connected, and we don't yet have an equally nice way to show that graphs are 3-connected. So we will cheat.

Suppose Q_3 had a vertex cut $\{u, v\}$ of size 2. Then in $Q_3 - u$, the vertex v would be a cut vertex: deleting it would bring us to $Q_3 - \{u, v\}$, which by assumption is not connected. We can prove that this doesn't happen by proving that for all vertices u, $Q_3 - u$ has no cut vertices: that it is

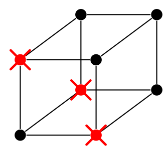
[1] This document comes from an archive of the Math 3322 course webpage: http://misha.fish/archive/3322-fall-2024

still 2-connected. The nice thing about Q_3 is that all vertices are identical—so we only have to do this once!

Here is an ear decomposition of $Q_3 - u$ for an arbitrary vertex u:



So $Q_3 - u$ is 2-connected, and therefore Q_3 is 3-connected. However, it does does have a 3-vertex cut:



Therefore $\kappa(Q_3)$ is exactly 3. $\square$

We disconnect the cube graph by deleting all the neighbors of a vertex. This works in general:

Proposition 1.2. *For any graph G, $\kappa(G) \leq \delta(G)$ (where $\delta(G)$ is the minimum degree of G.)*

Proof. Actually, there is one more thing to check here.

If v is a vertex of G with $\deg(v) = \delta(G)$, then deleting all the neighbors of v leaves a graph in which v is an isolated vertex, and this is *usually* a vertex cut: the result is not connected, provided any vertices other than v are left!

The only case to worry about is this: what if nothing other than v is left after we delete the neighbors of v? This happens when v is adjacent to every other vertex, and since v has the minimum degree in the graph, all vertices must be adjacent. In other words, $G = K_n$ for some n.

In this case, $\delta(G) = n - 1$, and we have defined $\kappa(G) = n - 1$ artificially. So the inequality $\kappa(G) \leq \delta(G)$ continues to hold in the exceptional case, too (which is one reason why we made that definition). $\square$

2 Local connectivity and $s - t$ cuts

2.1 A few more definitions

Now let's suppose we have a graph G and we pick two vertices s and t. An $s - t$ **cut** in G is a vertex cut U that separates s from t: $G - U$ must still contain both s and t, but they must be in different components. We define

$$\kappa_G(s,t) = \min\{|U| : U \text{ is an } s - t \text{ cut}\}.$$

When it is clear what G is, we drop the subscript and just write $\kappa(s,t)$.

If s and t are adjacent, then no $s-t$ cut can exist: you can't destroy the edge st by deleting vertices that aren't s or t. In that case, we say that $\kappa(s,t) = \infty$.

There are, broadly speaking, two reasons to care about the quantity $\kappa(s,t)$.

1. In some applications, you don't care whether the entire graph remains connected: you just have a "starting" and "ending" vertex that still need to communicate.

2. Finding $\kappa(s,t)$ can help us find $\kappa(G)$, and for several reasons, finding $\kappa(s,t)$ is much easier.

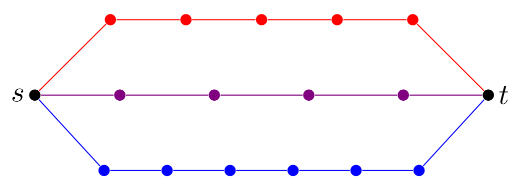
   We will see one of those reasons today and in the next lecture: Menger's theorem. The other reason is more properly discussed in a linear programming class: there are good algorithms for computing $\kappa(s,t)$ using linear programming techniques.

The connection between $\kappa(s,t)$ and $\kappa(G)$ is that $\kappa(G)$ is the minimum of $\kappa(s,t)$ over all pairs of non-adjacent vertices (s,t). (An exception to this is the complete graph K_n, in which there are no such pairs, and we have a separate definition of $\kappa(G)$.)

2.2 Internally disjoint paths

As with many other quantities in graph theory, $\kappa(s,t)$ is the result of an optimization problem, so it is easy to bound from one direction and hard to bound from the other. If you find an $s-t$ cut U, then you know that $\kappa(s,t) \geq |U|$. But how do we prove a lower bound?

One answer is the following. Suppose G contains a subgraph that looks like the diagram below:



Then it is clear that at least three vertices need to be deleted to disconnect s from t. One of the red vertices must be deleted to destroy the red $s-t$ path. One of the purple vertices must be deleted to destroy the purple $s-t$ path. One of the blue vertices must be deleted to destroy the blue $s-t$ path. Therefore $\kappa(s,t) \geq 3$. (In the subgraph, we have $\kappa(s,t) = 3$; if G has other vertices and edges, the value of $\kappa_G(s,t)$ is not as clear.)

In general, a collection of $s-t$ paths is **internally disjoint** if no two paths share any common vertices other than s and t. That is what is needed for this argument to work. If two paths have a common internal vertex, deleting that vertex destroys two paths at once. On the other hand, if we can find a collection of k internally disjoint $s-t$ paths, then we know $\kappa(s,t) \geq k$.

The big theorem we will end the semester with is:

Theorem 2.1 (Menger)**.** *If s,t are two non-adjacent vertices in a graph G, then we can find a collection of $\kappa(s,t)$ internally disjoint $s-t$ paths.*

In other words: the lower bound we find here always matches the upper bound!

The theorem about cycles in 2-connected graphs is a quick corollary of Menger's theorem.

Corollary 2.2. *If G is a 2-connected graph, then any two vertices in G lie on a common cycle.*

Proof. Suppose we take two vertices $s, t \in V(G)$ that are not adjacent. Then $\kappa(s,t) \geq 2$ because $\kappa(G) \geq 2$: since G has no 1-vertex cuts, in particular it has no 1-vertex cuts separating s and t. Therefore by Menger's theorem, there are two internally disjoint $s - t$ paths in G. Put them together, and you have a cycle containing s and t.

When s and t are adjacent, we need a separate argument. Pick a third vertex u adjacent to at least one of s or t (in a 2-connected graph, there must be at least 3 vertices, and s, t cannot be their own connected component, so there must be such a u). Then:

- If u is adjacent to both s and t, then (u, s, t, u) is the cycle we want.
- If u is adjacent to only one of them—say, s but not t—then find two internally disjoint $u - t$ paths. One of them does not contain s: combine that path $(u, v_1, v_2, \dots, v_k, t)$ with the path (t, s, u) to get a cycle containing s and t.

In all cases, we get the cycle we want, proving the theorem. □

But we will continue in a state of "proof debt", because we will not prove Menger's theorem in this lecture; we will wait until the next lecture.

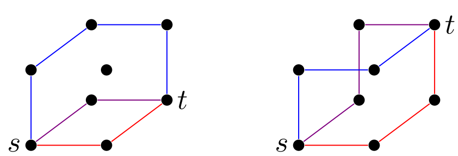
2.3 Applications

We can apply the idea of internally disjoint paths to determine $\kappa(G)$ as well. For instance, we can show $\kappa(Q_3) = 3$ by a second argument that does not involve ear decompositions.

Claim 2.3. *If G is the cube graph, and s, t are any two non-adjacent vertices of G, then $\kappa(s,t) \geq 3$. As a result, $\kappa(G) = 3$.*

Proof. It's enough to consider two cases: taking $s = (0,0,0)$ and $t = (0,1,1)$, or taking $s = (0,0,0)$ and $t = (1,1,1)$. That's because the cube graph has many automorphisms. In particular, it has an automorphism taking any pair of non-adjacent vertices to one of these two pairs.

In each case, we prove $\kappa(s,t) \geq 3$ by finding three internally disjoint $s - t$ paths:



Therefore $\kappa(s,t) \geq 3$ in both cases. As a result, $\kappa(G) \geq 3$.

As before, $\kappa(G) \leq 3$ because we can delete the 3 neighbors of a vertex. □

We were able to save ourselves a lot of work here by exploiting the symmetry of the cube. In general, we may need to consider more cases. (But see the practice problems for a way to reduce their number.)

3 Practice problems

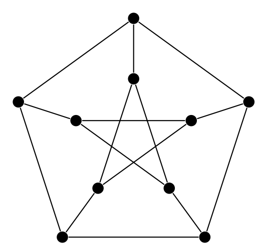
1. Draw the hypercube graph Q_4. Find four internally disjoint $s-t$ paths in three cases:
   (a) s and t are two vertices at distance 2 from each other.
   (b) s and t are two vertices at distance 3 from each other.
   (c) s and t are two opposite vertices.

   Conclude that $\kappa(Q_4) = 4$.

2. Use induction on n to prove that if s, t are two non-adjacent vertices in Q_k, then $\kappa(s,t) \geq k$. There are two cases here:
   - If s and t are two opposite vertices, you should be able to write down paths directly.
   - If s and t are not opposite vertices, then they're in the same copy of Q_{k-1}. Apply the inductive hypothesis to find $k-1$ paths, then find one more path guaranteed to be internally disjoint from the previous ones.

   Conclude that $\kappa(Q_k) = k$.

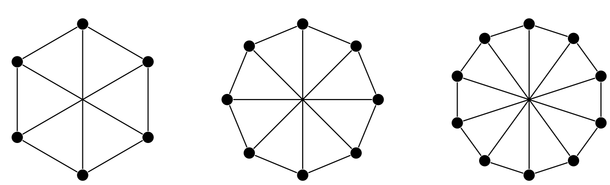
3. Let G be the Petersen graph, shown below:

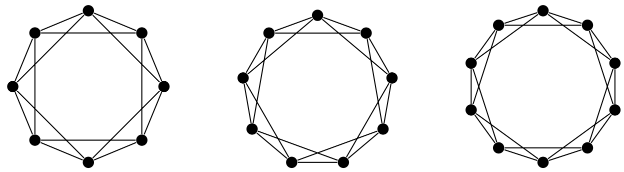
   Pick any two non-adjacent vertices in this graph, and find three internally disjoint paths between them.

   (Note: the definition of the Petersen graph is symmetric enough that solving this problem for one pair of non-adjacent vertices is enough to know that it has a solution for all such pairs. Therefore you have just shown that the Petersen graph is 3-connected.)

4. The Harary graphs, which we saw in lecture 7 as an example of regular graphs, are actually more important than that. Whenever $0 \leq r \leq n-1$ and at least one of r and n is even, the Harary graph $H_{n,r}$ is an example of an r-regular graph with $\kappa(H_{n,r}) = r$. (Since all r-connected graphs have minimum degree at least r, this means that the Harary graphs achieve their connectivity with as few edges as possible.)
   (a) Prove this when $r = 3$. The Harary graphs $H_{6,3}$, $H_{8,3}$, and $H_{10,3}$ are shown as examples below:

(b) Prove this when $r = 4$. The Harary graphs $H_{8,4}$, $H_{9,4}$, and $H_{10,4}$ are shown as examples below:



(c) If you are now feeling confident, prove that $\kappa(H_{n,r}) = r$ for all r.

(Note: in both parts of this problem, reasoning directly from the definition and using Menger's theorem are both viable approaches.)

5. A graph is called **fragile** if it has a vertex cut which is an independent set.

   Prove that all connected triangle-free graphs on at least 3 vertices are fragile.

   (Source: "A note on fragile graphs" by Guantao Chen and Xingxing Yu.)

6. Let G be a k-connected graph, and let H be a new graph built from G by adding a new vertex v and making it adjacent to k of the vertices of G.

   (a) Reasoning directly from the definition, prove that H is also k-connected.

   (b) If s is a vertex of G and T is a set of vertices, then an $s - T$ **fan** consists of an $s - t$ path for each $t \in T$, such that the paths share no vertices other than s.

   Use Menger's theorem and the previous part of this problem to prove the following:

   Theorem 3.1. *If G is a k-connected graph, s is a vertex of G, and $T \subseteq V(G)$ with $s \notin T$ and $|T| = k$, then G contains an $s - T$ fan.*

7. Let v be an arbitrary vertex of a graph G (not a complete graph) and suppose that the following is true:

   - For every vertex w not adjacent to v, $\kappa(v, w) \geq k$.
   - For every two vertices x, y that are adjacent to v but not each other, $\kappa(x, y) \geq k$.

   Prove that $\kappa(G) \geq k$.

   (This is one way to find $\kappa(G)$ by checking $\kappa(s, t)$ for relatively few pairs (s, t).)