Math 3322: Graph Theory[1] Mikhail Lavrov

Lecture 7: Regular graphs

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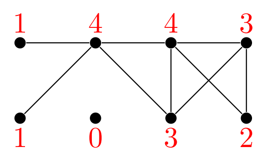
1 Degree sequences and the graphic sequence problem

The **degree sequence** of a graph G is a sequence of numbers that gives all the degrees of all the vertices of G.

Despite the word "sequence", these don't come in any particular order, because the vertices of a graph don't have a particular order. (There might be a natural order you're tempted to use based on the way we drew a graph, but we don't want the properties of graphs to depend on how we draw them!) Technically, we should be talking about the "degree multiset" of a graph.

To avoid the temptation of looking for meaning in the order of the degree sequence, we typically list the degrees in sorted order. (We'll stick to descending order in this lecture.) For example, we'd write the degree sequence of the graph below as $4, 4, 3, 3, 2, 1, 1, 0$.



The basic question we'll tackle today and in the next lecture is the following: given a sequence of numbers, can you tell if it's the degree sequence of a graph? We call such a sequence a **graphic sequence**. For example, $4, 4, 3, 3, 2, 1, 1, 0$ is a graphic sequence, and the graph above is proof. (There are other graphs with the same degree sequence, too.)

We already have some basic tests to use to answer this question:

- In a graphic sequence of length n, every term must be an integer between 0 and $n - 1$.
- The sum of the terms in a graphic sequence must be even. To put it another way, the number of odd terms must be even.

But there are also non-graphic sequences out there that pass both tests. For example, the sequence $4, 3, 2, 1, 0$ is not a graphic sequence, even though all 5 terms are between 0 and 4, and exactly 2 of them are odd. (Think about why.)

2 Regular graphs

A **regular graph** is a graph in which every vertex has the same degree. More specifically, an r-**regular graph** is a graph in which every vertex has degree r. The degree sequence of such a graph is $r, r, r, \ldots, r, r$.

[1] This document comes from an archive of the Math 3322 course webpage: http://misha.fish/archive/3322-fall-2024

We'll begin looking at the graphic sequence problem with regular graphs, because there is a nice answer to it in those cases:

Theorem 2.1. *An r-regular graph on n vertices exists whenever $0 \le r \le n-1$ and at least one of r or n is even.*

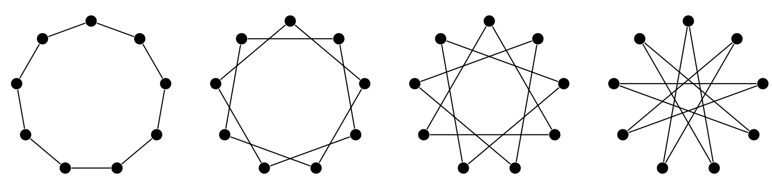
In other words: out of all sequences of the form $r, r, \ldots, r$, the graphic sequences are exactly the ones that pass both of our "basic tests" from the previous section!

Proof. To prove that an r-regular graph on n vertices exists, all we have to do is construct one.

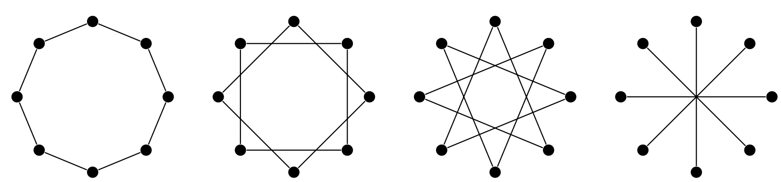
For most values of r and n, there are many many options available. We just have to show how to construct one of them. The hard part about this proof is that we have to give a general rule based on the values of r and n. (For specific values of r and n, the proof would be very short and just look like "here, look at the graph I drew.")

Our recipe for constructing an r-regular graph on n vertices begins by constructing several "ingredient" graphs. These will all have n vertices called $v_1, v_2, \ldots, v_n$, and we will draw them spaced evenly around a circle.

The "ingredient" graph $G_{n,k}$ is defined to be the graph with an edge between every pair of vertices that are "k steps apart" around the circle. Formally, it has an edge $v_i v_j$ whenever $i - j \equiv \pm k \bmod n$. Here are the graphs $G_{9,1}$, $G_{9,2}$, $G_{9,3}$, and $G_{9,4}$, for illustration:



When n is odd, each of these "ingredient" graphs will be 2-regular. When n is even, there is one exception: the graph $G_{n,n/2}$ is a 1-regular graph. Here's what $G_{8,1}$, $G_{8,2}$, $G_{8,3}$, and $G_{8,4}$ look like:



Another important property of these graphs is that even though they are defined on the same set of vertices, they share no edges at all. Each possible edge $v_i v_j$ appears in exactly one of the "ingredient" graphs.

Now that these "ingredients" are defined, here is the rule for how we can put them together to get an r-regular graph on n vertices:

- If r is even, take the union $G_{n,1} \cup G_{n,2} \cup \cdots \cup G_{n,r/2}$.

  Each vertex v_i has 2 incident edges in each of the $r/2$ graphs in the union, and all of these incident edges are different. In total, $\deg(v_i) = 2 + 2 + \cdots + 2 + 2 = 2 \cdot \frac{r}{2} = r$.

- If r is odd and n is even, take the union $G_{n,1} \cup G_{n,2} \cup \cdots \cup G_{n,(r-1)/2} \cup G_{n,n/2}$.

  Each vertex v_i has 2 incident edges in the first $(r-1)/2$ graphs in the union, and 1 incident edge in the last graph. Again, all of these edges are different. In total, $\deg(v_i) = 2+2+\cdots+2+2+1 = 2 \cdot \frac{r-1}{2} + 1 = r$.

There are some special cases that are maybe technically covered by the bullet points above, but are worth mentioning separately.

When $r = 0$, there's nothing to take the union of. In this case, we can just take the empty graph on n vertices.

When $r = 1$ (and n is even), $\frac{r-1}{2}$ is 0, so we don't include any of the 2-regular ingredient graphs, and only take $G_{n,n/2}$.

When $r = n - 1$, we'll end up taking the union of all the ingredient graphs we had, getting the complete graph K_n.

Anyway, this construction covers all the cases we needed to prove the theorem for. Now we know that all of these graphs exist! □

The graphs we construct in the proof of this theorem have a special name. They are called the **Harary graphs**. You sometimes see $H_{n,r}$ used to denote the r-regular Harary graph on n vertices. (Unlike the notation $G_{n,k}$ from our proof, which has no existence outside of today's lecture notes, the notation $H_{n,r}$ is at least a little bit standard.)

3 A recipe for realizing a degree sequence

In general, if you are testing to see if a sequence is graphical, there's two ways this can go down:

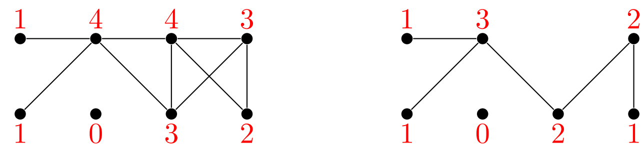
- If the sequence is not graphical, you'd like to point to some test it fails, but that all graphical sequences should pass.
- If the sequence is graphical, you'd like to draw a graph with that degree sequence, and then point to it and say "Voilà!".

  (This is called "realizing" the degree sequence. Not "realize" in the sense of "huh, I didn't think of that until just now" but "realize" in the sense of "make something real".)

Today, we'll talk about a general algorithm for doing the second thing. Sometimes, the algorithm will work, and then we'll get a graph with the degree sequence we wanted. Sometimes, the algorithm will not work...

...which is less disappointing than it could be, because in the next lecture, we'll prove that the algorithm works for all graphic sequences. So the algorithm is also a test: if it fails, then we know that the sequence is not graphic! (But we won't prove this today.)

The motivation is this. If we delete one of the degree-4 vertices of our graph with degree sequence $4, 4, 3, 3, 2, 1, 1, 0$, we get a graph with degree sequence $3, 2, 2, 1, 1, 1, 0$. Constructing a graph with the shorter degree sequence will be easier, and then we can just add the degree-4 vertex back in.



In general, if you take a graph with vertices $v_1, v_2, \ldots, v_n$ and delete vertex v_1, what happens to the degree sequence? Well, first of all, the $\deg(v_1)$ term disappears. Second, every neighbor of v_1 has its degree decrease by 1.

We can't do this deletion in the same way when all we have is the degree sequence. It depends on what the neighbors of v_1 are. So we will make a guess about this:

Guess. *Suppose we have a sequence $d_1 \ge d_2 \ge \cdots \ge d_n$. We will make a guess that if there's a graph G with this degree sequence, then the vertex with degree d_1 is adjacent to the vertices of degrees $d_2, d_3, \ldots, d_{d_1+1}$.*

It will turn out later that this guess is justified. There might be many graphs with this degree sequence, and the guess might be false for some of them, but there will be at least one graph for which the guess is true. We will not prove that today, though.

If the guess is true, and we delete the vertex with degree d_1, then we get a graph with degree sequence

$$d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n.$$

(The nested subscripts might be confusing; essentially, we decrease the d_1 largest terms by 1.)

So our algorithm will do the following, given a sequence:

1. Sort the sequence so that $d_1 \ge d_2 \ge \cdots \ge d_n$.
2. Apply the transformation above, turning the sequence into

   $$d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n.$$

   Use this algorithm to solve the problem for this shorter sequence. (It might need to be sorted again.)
3. Take the graph we got, and add a new vertex of degree d_1, adjacent to the vertices with degree $d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1$.

At some point, we may end up with a sequence that's obviously not graphic: either $d_1 \ge n$ (in which case we can't subtract 1 from the next n terms, because there's not n terms that come after d_1) or $d_n < 0$ (due to subtraction). In that case, we give up.

In the next lecture, we'll see that we only need to give up in cases where the starting sequence was not graphic. This shouldn't be obvious yet: it relies on the guess we made being correct.

3.1 An example

Let's try this out on the sequence $3, 3, 2, 2, 2$.

The first thing we do is delete the first 3 and subtract 1 from the next three terms. This gives us $2, 1, 1, 2$, which we sort to get the sequence $2, 2, 1, 1$.

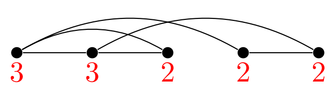
The second thing we do is delete the first 2 and subtract 1 from the next two terms. This gives us $1, 0, 1$, which we sort to get the sequence $1, 1, 0$.

Finally, we delete the first 1 and subtract 1 from the next term. This gives us $0, 0$. At this point, it is easy[2] to find a graph with this degree sequence. It's the graph with 2 vertices and no edges:

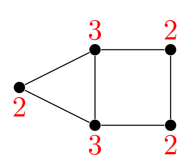
Now, we begin to work backwards. To get the degree sequence $1, 1, 0$, we add a new vertex to the left, and make it adjacent to the next vertex. It's convenient to then reorder the vertices in our drawing so that the degrees, going from left to right, match the sequence $1, 0, 1$ we had half a step earlier:

To get the degree sequence $2, 2, 1, 1$ from $1, 0, 1$, we add a vertex to the left, and make it adjacent to the next 2 vertices. Once again, we follow this up by rearranging our drawing so that the vertex degrees are $2, 1, 1, 2$ from left to right:

Finally, to get the degree sequence $3, 3, 2, 2, 2$ from $2, 1, 1, 2$, we add a vertex to the left, and make it adjacent to the next three vertices:



If you want to, you can follow up by rearranging the diagram to something more pleasing to the eye. (This step is purely optional.)



Note that this is only one of the possible graphs with this degree sequence. (Another graph with this degree sequence is the complete bipartite graph $K_{2,3}$.)

[2]Depending on your own threshold for "easy", you can stop and draw the graph earlier. Stop at any point when you feel confident that you can draw a graph with the current degree sequence!

4 Practice problems

1. Let's take a closer look at the Harary graphs and their "ingredient" graphs.

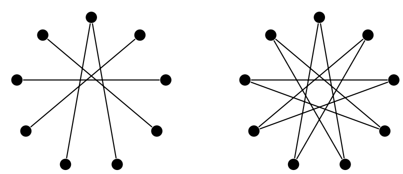
   (a) Draw a diagram of $H_{6,0}$ through $H_{6,5}$.

   (b) For which n and k is the "ingredient" graph $G_{n,k}$ connnected?

   (c) For which n and r is the Harary graph $H_{n,r}$ connected? (It might be more straightforward to answer the opposite question: when is $H_{n,r}$ *not* connected?)

2. If n and r are both odd, then an r-regular graph on n vertices does not exist, but the Harary graph $H_{n,r}$ is still defined! In this case, it is a graph on n vertices where $n-1$ of them have degree r, and one has degree $r+1$.

   We can prove this with a slight modification of the proof of Theorem 2.1. A new "ingredient" graph $G_{n,n/2}$ is required, which is nearly 1-regular, but has one vertex of degree 2. For example, here is the "ingredient" graph $G_{9,4.5}$ (on the left), next to a picture of the old "ingredient" graph $G_{9,4}$ that it replaces (on the right):

   (a) Construct a general "ingredient" graph $G_{n,n/2}$ for all odd n. This should not just be a graph with the correct degree sequence; it should also fit in nicely with the other "ingredient" graphs.

   (b) Use this to prove that when r and n are both odd, a Harary graph $H_{n,r}$ exists: a graph on n vertices where $n-1$ of them have degree r, and one has degree $r+1$.

3. Our algorithm makes the "guess" that the highest degree vertex in a graph should be adjacent to the vertices with the next highest degrees. Of course, this is not always necessarily true.

   But it's also not a completely random guess; sometimes it is the only possibility! For example, convince yourself that in a graph with degree sequence $3, 3, 3, 1, 1, 1$, each of the degree-3 vertices **must** be adjacent to both other degree-3 vertices.

   (How much can you generalize this example?)

4. Using our algorithm or otherwise, check that none of the sequences below are graphic:

   (a) $7, 6, 5, 4, 3, 2, 1$.

   (b) $4, 4, 3, 3, 3, 2, 2$.

   (c) $6, 6, 5, 3, 2, 2, 1, 1$.

   (d) $n-1, n-1, n-1, n-1, \underbrace{3, 3, \ldots, 3}_{n-4 \text{ times}}$ (for any n).

5. Use our degree realization algorithm to construct a graph with degree sequence $4, 3, 2, 2, 1$.

6. Here is a fragment of an alternate proof of the $r = 3$ case of Theorem 2.1.

   ...assume that a 3-regular graph H on $n - 4$ vertices exists. Then, we can create a 3-regular graph G on n vertices, just by adding a new connected component to H: four new vertices adjacent to each other and to no other vertices ...

   What kind of a proof is this? What else do we need to do to finish the proof of Theorem 2.1 using this idea?

7. Theorem 2.1 tells us when an r-regular, n-vertex graph exists. In this practice problem, we'll look at when a *bipartite* r-regular graph exists.

   (a) Prove that if G is a bipartite r-regular graph with bipartition (A, B), then $|A| = |B|$.

   (Hint: A practice problem from the previous lecture might help here.)

   (b) Imitate the style of Theorem 2.1 to prove that a bipartite r-regular graph with $2n$ vertices (n on each side of the bipartition) exists whenever $0 \leq r \leq n$.