Math 3322: Graph Theory[1] Mikhail Lavrov

Lecture 9: Graph isomorphisms

September 10, 2024 *Kennesaw State University*

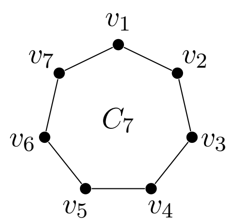
1 Isomorphic graphs

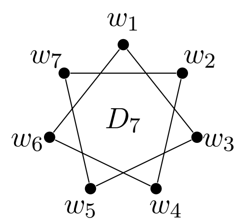
1.1 Two isomorphic families

As a motivating example, consider the following two families of graphs. (The first is defined for all $n \geq 3$, the second for all $n \geq 5$.)

- Let C_n be the cycle graph, with vertices $v_1, v_2, \ldots, v_n$ and edges $v_i v_{i+1}$ for $i = 1, \ldots, n-1$, as well as $v_n v_1$.
- Let D_n be the graph with vertices $w_1, w_2, \ldots, w_n$ and edges $w_i w_{i+2}$ for $i = 1, \ldots, n-2$, as well as $w_{n-1} w_1$ and $w_n w_2$. (As far as I know, neither this graph nor the notation D_n is relevant outside this example.)

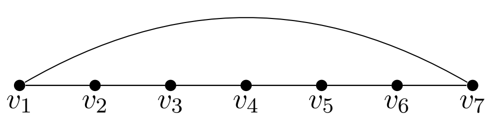
Here is an example of each graph:

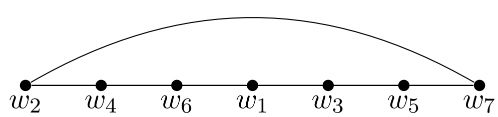




Are C_7 and D_7 the same graph? Well, definitely not: they don't even have the same set of vertices. Even if we did change all the w's to v's, the graphs still wouldn't be the same: they would have different sets of edges.

At the same time, if we look at these two graphs the right way, they have the same structure. The following diagrams reveal the similarity:





Formally, we say that these two graphs are isomorphic.

We define a **graph isomorphism** from graph G to graph H to be a function $f\colon V(G) \to V(H)$ with two properties:

1. f is a bijection—for every vertex $y \in V(H)$, there is exactly one vertex $x \in V(G)$ such that $f(x) = y$.

[1]This document comes from an archive of the Math 3322 course webpage: `http://misha.fish/archive/3322-fall-2024`

2. f preserves adjacency—for any two vertices $v, w \in V(G)$, v and w are adjacent in G if and only if $f(v)$ and $f(w)$ are adjacent in H.

Two graphs are **isomorphic** if there is an isomorphism from one graph to the other. In which direction? It doesn't matter: because we defined f to be a bijection in the definition above, it has an inverse $f^{-1}\colon V(H) \to V(G)$, and we can check that f^{-1} is an isomorphism from H to G.

In this example, the two identical-looking diagrams correspond to the following isomorphism from C_7 to D_7:

x	v_1	v_2	v_3	v_4	v_5	v_6	v_7
$f(x)$	w_2	w_4	w_6	w_1	w_3	w_5	w_7

(This table is a clearer way to write out that $f(v_1) = w_2$, $f(v_2) = w_4$, and so on.)

Intuitively speaking, two graphs are isomorphic if they're "the same graph, but with different names for the vertices". The graph isomorphism is a dictionary that translates between vertex names in G and vertex names in H.

We can say more:

Claim 1.1. *For all odd $n \geq 3$, C_n and D_n are isomorphic.*

Proof. A proof-writing note, first: to prove the general claim, we should find a general rule for an isomorphism from C_n to D_n. Typically, this is done by looking at small examples and generalizing, but it's good to look for options that will generalize well. In the case of C_7 and D_7, there's 14 possible automorphisms and some are easier to generalize than others.

The automorphism in the example above, however, was chosen to be the one that generalizes the best. Its generalization is given by the following function from $\{v_1, \dots, v_n\}$ to $\{w_1, \dots, w_n\}$:

$$f(v_i) = \begin{cases} w_{2i} & \text{if } i < n/2 \\ w_{2i-n} & \text{if } i > n/2. \end{cases}$$

This is a bijection: the first case of the definition always gives us a vertex w_j where j is even, and the second case always gives us a vertex w_j where j is odd, so for any j, we know which case to look at to find an inverse $f^{-1}(w_j)$.

At this point, working directly from the definition, we should be checking for *every* pair of vertices in C_n whether f preserves their relationship in D_n. There is a common shortcut, however. First, we observe that C_n and D_n both have n edges. This means that if we check that f takes edges of C_n to edges of D_n, we'll be done; this "uses up" all n edges of D_n, so f will also take non-edges of C_n to non-edges of D_n.

Let's go through all the edges of C_n in four cases:

1. First, consider edges $v_i v_{i+1}$ where $i + 1 < n/2$. We get $f(v_i) = w_{2i}$ and $f(v_{i+1}) = w_{2i+2}$, which are indeed adjacent in D_n.
2. Second, consider edges $v_i v_{i+1}$ where $i > n/2$. We get $f(v_i) = w_{2i-n}$ and $f(v_{i+1}) = w_{2i+2-n}$, which are also adjacent in D_n.

3. One special case we've missed is $v_n v_1$, which doesn't follow the pattern. We get $f(v_n) = w_n$ and $f(v_1) = w_2$, and these are adjacent in D_n.
4. The other special case is $v_i v_{i+1}$ for $i = \frac{n-1}{2}$ and $i+1 = \frac{n+1}{2}$, because here, different cases of f apply to v_i and v_{i+1}. We get $f(v_i) = w_{2i} = w_{n-1}$ and $f(v_{i+1}) = w_{2i+2-n} = w_1$, and these are adjacent in D_n.

In all cases, vertices adjacent in C_n are sent to vertices adjacent in D_n by f, completing our check that f is an isomorphism. □

1.2 Non-isomorphic graphs

Why did we limit our proof to odd n? Well, because C_n is not isomorphic to D_n when n is even!

This seems hard to check from the definition, though. To prove than an isomorphism does not exist directly, we'd have to rule out all $n!$ possible functions from $V(C_n)$ to $V(D_n)$. This is hard to do even when we look at C_8 and D_8, for example.

There is, however, a short proof.

Claim 1.2. *For all even $n \geq 6$, C_n is not isomorphic to D_n.*

Proof. The graph C_n is connected: for any v_i and v_j, if $i < j$, then the path $(v_i, v_{i+1}, \dots, v_j)$ connects them, and if $i > j$, just reverse the path from v_j to v_i.

The graph D_n is not connected when n is even: there is no edge from any of the vertices

$$\{w_1, w_3, w_5, \dots, w_{n-1}\}$$

to any of the vertices

$$\{w_2, w_4, w_6, \dots, w_n\},$$

so there is no path from a vertex in the first set to a vertex in the second set. □

The implied rule is: when two graphs are isomorphic and one is connected, so is the other, so checking that C_n is connected and D_n isn't proves that they can't be isomorphic. This is something we can prove directly: if f is an isomorphism from G to H, and G is connected, then we can use f to turn any path in G into a path in H, letting us find a path in H between any two vertices.

It's also a general principle, though. *Any* property that depends only on the structure of a graph—not on, say, the way the vertices are labeled, or the way that they're ordered, or the way that the graph is drawn—should be preserved by isomorphism. For example, if G is isomorphic to H, then we can say that:

- G and H have the same number of vertices and edges.
- G and H have the same number of connected components.
- G is bipartite if and only if H is bipartite.
- G and H have the same diameter.

- G and H have the same degree sequence; in particular, the same minimum degree and the same maximum degree. We can be more precise: if f is a graph isomorphism from G to H, and v is a vertex of G, then $\deg_G(v) = \deg_H(f(v))$.

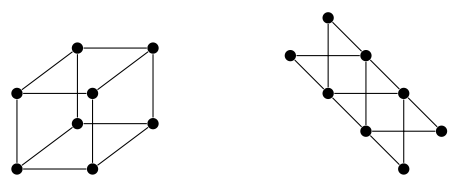
Sometimes, we call these **invariants** of a graph: "invariant" because they not change when we rearrange the vertices.

Watch out for one common pitfall. Even if G and H share some vertices, the isomorphism between G and H does not have to care about the shared vertices—isomorphisms don't care about vertex names. If G and H both have a vertex v, and are isomorphic, v might do different things in the two graphs (for example, $\deg_G(v)$ might be different from $\deg_H(v)$).

However, if you can describe some property without making reference to vertex names, then it should be preserved by isomorphism. For example, if G has the property "no two vertices of degree 4 in G are adjacent", and H is isomorphic to G, then H must also have this property.

1.3 Isomorphic or not?

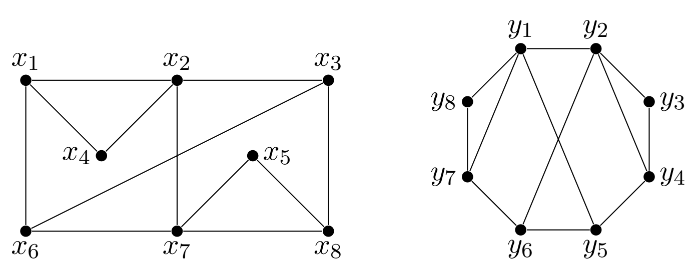
Identifying graph invariants is one of the best ways in general to prove that two graphs are not isomorphic. Here are a few examples to look at:



The middle graph here immediately sticks out as different from the other two by its degree sequence. It has vertices of degrees 2 and 4. The other two graphs are 3-regular.

The first graph—the cube graph Q_3—is bipartite. In the third graph, there are odd cycles (for example, going around the top half of the octagon, and then coming back along the horizontal diagonal, gives a cycle of length 5). Therefore the first and third graph are also not isomorphic.

Finding invariants like this can also help us along even when two graphs *are* isomorphic. Consider the following two graphs as an example:



These graphs—I promise you—are isomorphic. Can we find the isomorphism?

A good place to start is to label the vertices by degree. On the left, x_2 and x_7 have degree 4, x_4 and x_5 have degree 2, and the rest have degree 3. On the right, y_1 and y_2 have degree 4, y_3 and y_8 have degree 2, and the rest have degree 3. The isomorphism must preserve these vertex degrees: for example, we must set $f(x_2)$ to either y_1 or y_2.

Which one? Normally, we'd try both and see. In this case, there's a nice time-saving observation that can help us. The diagram of the graph on the right is symmetric: if you take its mirror image, flipping it horizontally, you get an identical diagram. The topic of symmetry is one we'll return to in a bit, but in this case it tells us that we have two isomorphisms: from one isomorphism, we can get another by applying mirror symmetry. If one isomorphism sends x_2 to y_1, the other will send x_2 to y_2, which means that we can make the choice arbitrarily and not have to worry about making a mistake.

Let's decide to set $f(x_2) = y_1$, which means $f(x_7) = y_2$. Does this mean that we get to make another arbitrary choice for x_4 and x_5? No: our hand is forced! Now that we've made some initial decisions, we can distinguish x_4 as the degree-2 vertex adjacent to x_2, which means that $f(x_4)$ must be the degree-2 vertex adjacent to $f(x_2)$ or y_1. Therefore $f(x_4)$ can only be y_8, and $f(x_5)$ is left to be y_3.

Similarly, our decisions about the degree-3 vertices are forced. For each of them, just look at its neighbors *among the vertices we've already placed*. Let's start with x_1. It is adjacent to x_2 and x_4, so $f(x_1)$ is adjacent to $f(x_2) = y_1$ and $f(x_4) = y_8$. The only such vertex is y_7 and so $f(x_1) = y_7$. I'll leave you to check, by similar reasoning, that $f(x_3) = y_5$, $f(x_6) = y_6$, and $f(x_8) = y_4$ are are only options.

We can be certain that the result is an automorphism if we check every adjacency along the way. (For example, when we get to x_8, it is adjacent to x_3, x_5, and x_7, and so we should check that $f(x_8)$ is adjacent to $f(x_3)$, $f(x_5)$, and $f(x_7)$.) If there are some we didn't check—and maybe in general, just to be safe—we should now verify that f satisfies the definition of an isomorphism. As before, we can count edges and take a shortcut: both graphs have 12 edges, so once we check that every edge on the left is sent to an edge on the right, we know that f is an isomorphism.

2 Fancier topics

2.1 Graph automorphisms (symmetries)

An **automorphism** of a graph G is an isomorphism from G to itself. The function $f\colon V(G) \to V(G)$ such that $f(v) = v$ for all vertices v is always an automorphism: the **identity** or **trivial** automorphism. However, there may be more complicated ones. For example, the path graph P_n has an automorphism that "reverses the path": $f(v_i) = v_{n+1-i}$. We saw another one in the last example in the previous section: the diagram on the right had mirror-image symmetry, which can be described by an automorphism that swaps the pairs $y_1 \leftrightarrow y_2$, $y_8 \leftrightarrow y_3$, $y_7 \leftrightarrow y_4$, and $y_6 \leftrightarrow y_5$.

Just as in that example, automorphisms are useful because they let us describe ways in which a graph is "symmetric": either geometrically or abstractly. This lets us avoid dealing with many identical cases in proofs about that graph.

Here's an example. First, we will prove a lemma about automorphisms of the cycle graph C_n, which admittedly takes a bit of work.

Lemma 2.1. *If x and y are any two adjacent vertices of the cycle graph C_n, then C_n has an automorphism f such that $f(x) = v_1$ and $f(y) = v_n$.*

Proof. If g and h are two automorphisms of C_n, then their composition $g \circ h$ is also an automorphism. So we will build the automorphism f we want by composing simpler automorphisms.

One simple automorphism of C_n is the "left shift" automorphism s, defined by

$$s(v_i) = \begin{cases} v_{i-1} & \text{if } i > 1, \\ v_n & \text{if } i = 1. \end{cases}$$

If it happens that $x = v_k$ and $y = v_{k-1}$ for some $k \geq 2$, then applying the left shift automorphism $k-1$ times will take x to v_1 and y to v_n. (Also, if $x = v_1$ and $y = v_n$ already, then the identity automorphism is the automorphism we're looking for.)

However, x and y might appear in the other order around the cycle. In this case, we'll need another simple automorphism of C_n: the "reverse" automorphism r, defined by $r(v_i) = v_{n+1-i}$ for all i.

This is the automorphism we're looking for if $x = v_n$ and $y = v_1$. Finally, if $x = v_{k-1}$ and $y = v_k$ for some $k \geq 2$, then we can apply the left shift automorphism $k-1$ times (taking x to v_n and y to v_1) then apply the reverse automorphism. □

Now we can use this lemma to make lots of proofs about C_n much easier, because we don't have to check many very similar cases! For example:

Theorem 2.2. *If xy is any edge of C_n, then $C_n - xy$ (the graph we obtain by deleting xy) is still connected.*

Proof. By Lemma 2.1, we may assume that $x = v_1$ and $y = v_n$. Here's how: let f be the automorphism that takes x to v_1 and $y = v_n$. Then the same f is also an isomorphism between $C_n - xy$ and $C_n - v_1 v_n$. So if we show that $C_n - v_1 v_n$ is connected, we conclude that $C_n - xy$ is connected.

(Warning: you may see proofs that skip the explanation of how the automorphism helps us, and just say "because the graph has such-and-such symmetry, we may assume this-and-that".)

When $x = v_1$ and $y = v_n$, then $C_n - xy = P_n$, which we already know is connected. (See lecture 3.) So $C_n - xy$ is connected for all edges xy. □

2.2 Copies of a graph

We often say, somewhat informally, that a graph G "contains a copy of H" for some smaller graph H. The formal meaning of this is that G has a subgraph which is isomorphic to H.

Such statements are very useful because they let us pick out small "patterns" inside a big graph. We will see this in action many times in this course. For now, let's just go back and talk about a few places we could already have made use of this notion.

We think of paths and cycles as special kinds of closed walks. However, we also have the path graph P_n and the cycle graph C_n. There is a connection:

- A path $(x_1, x_2, \dots, x_n)$ in a graph G can also be described as a copy of P_n inside G: the copy of P_n with vertices $x_1, x_2, \dots, x_n$ and edges $x_1x_2, \dots, x_{n-1}x_n$.

- Similarly, a cycle of length n in a graph G can also be described as a copy of C_n inside G.

Sometimes one way of thinking about these objects is more convenient than the other. Thinking of a path or cycle as a sequence of vertices means that we pick a starting vertex, an ending vertex (the same as the start, in the case of the cycle) and a direction of travel. So, for example, an $x-y$ path and a $y-x$ path are different objects from this point of view, even though they correspond to the same subgraph. We might sometimes prefer not to make this distinction—for example, if we want to know the *number of cycles* in a graph, it's probably better to count them as subgraphs, which avoids redundancy.

We also saw this in action when talking about induction on the hypercube Q_n. We can say, informally, that "Q_n consists of two disjoint copies of Q_{n-1} with corresponding vertices in the two copies joined by edges." What does this mean?

- Well, we start with two disjoint graphs that are each isomorphic to Q_{n-1}.

  (In the definition of Q_n, we can be more precise about what these are. Recall that Q_{n-1} has vertex set $\{0,1\}^{n-1}$: sequences of $n-1$ zeroes and ones. To make one copy of Q_{n-1}, we rename the vertices by adding a 0 to the end of each sequence. To make a second copy of Q_{n-1} disjoint from the first, we rename the vertices by adding a 1 at the end of each sequence instead.)

- Because they are both isomorphic to Q_{n-1}, they are isomorphic to each other. Let f be an isomorphism between them.

  (Specifically, we can say that f is the function which takes vertex $(x_1, \dots, x_{n-1}, 0)$ in one copy to the vertex $(x_1, \dots, x_{n-1}, 1)$ in the other copy.)

- For every vertex v in one copy of Q_{n-1}, we add an edge between v and $f(v)$: these are the edges between "corresponding vertices".

  (If we follow the specific details in these parenthetical comments, we'll get exactly the Q_n with vertex set $\{0,1\}^n$ and edges between vertices that differ in one coordinate. If we don't get that specific about the details of the copies of Q_{n-1} and the isomorphism between them, we'll still get a result isomorphic to Q_n.)

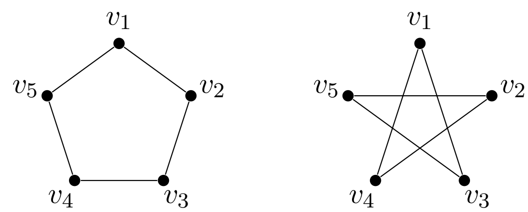
2.3 Self-complementary graphs (optional)

This section of the notes is a look at an interesting self-contained puzzle about graphs, which we now have the language to explore.

Define a graph G to be **self-complementary** if G is isomorphic to $\overline{G}$, the complement of G. For example, the path graph P_4 is self-complementary:

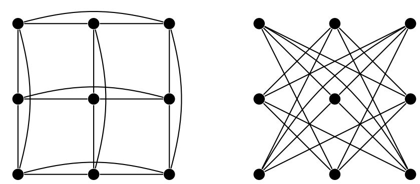
One possible isomorphism from the graph on the left to the graph on the right is given by $f(v_1) = v_2$, $f(v_2) = v_4$, $f(v_3) = v_1$, $f(v_4) = v_3$.

If you look for self-complementary graphs on 5 vertices, you'll probably find the 5-cycle:



There are many isomorphisms here (since C_5 has many automorphisms). One of them is given by $f(v_1) = v_1$, $f(v_2) = v_3$, $f(v_3) = v_5$, $f(v_4) = v_2$, and $f(v_3) = v_4$.

A more complicated example is a graph on 9 vertices known as the 3×3 "rook graph". Here, if we imagine the vertices as being laid out in a 3×3 grid, we declare vertices to be adjacent if they are in the same row or the same column. (The name "rook graph" comes from the way that a rook moves in chess.) In the diagram below, the rook graph as we just defined it is shown on the left, and its complement is shown on the right:



Can you find an isomorphism between these two graphs? *(Hint: you can start by declaring that the center vertex on the left maps to the center vertex on the right. From there, a lot of your decisions will be forced.)*

We can say a lot about the structure of a self-complementary graph. For example, if G is an n-vertex graph with m edges, then $\overline{G}$ has $\binom{n}{2} - m$ edges. If we want G to be isomorphic to $\overline{G}$, then we want $m = \binom{n}{2} - m$, which means $m = \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}$.

What's more, if G is self-complementary, its degree sequence must be symmetric. Suppose G has a vertex of degree k; then that same vertex has degree $n - 1 - k$ in $\overline{G}$. This means there must also be a vertex of degree $n - 1 - k$ in G: in fact, as many of these vertices as there are vertices of degree k. We see this in P_4, which has degree sequence $2, 2, 1, 1$. The other two examples we have take the easy way out by being regular of degree $\frac{n-1}{2}$. But from the degree sequence point of view, it's possible that another 5-vertex self-complementary graph exists, with degree sequence $3, 3, 2, 1, 1$. (In fact, there is such a graph; can you find it?)

When we come up with a new definition, and start playing around with it, there is the danger of putting in a lot of work for nothing. What if we spend hours and hours proving theorems about self-complementary graphs, only to find out that actually, we've found all the ones that exist, and there are no more? This would be very disappointing, so maybe it's best to first ask: are there actually many self-complementary graphs?

From what we've done, we can actually prove a limitation on their existence.

Proposition 2.3. *A self-complementary graph on n vertices can only exist if n has the form $4k$ or $4k + 1$ for some integer k.*

Proof. We know that a self-complementary graph on n vertices must have $\frac{n(n-1)}{4}$ edges, so in particular, $\frac{n(n-1)}{4}$ must be an integer. One of n or $n-1$ is always odd, so the other one must be divisible by 4. If n is divisible by 4, then $n = 4k$ for some integer k; if $n-1$ is divisible by 4, then $n = 4k+1$ for some integer k. □

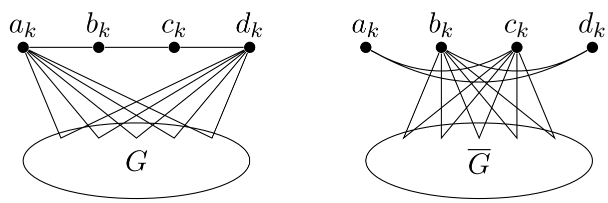
However, beyond that, there are no restrictions! Here's the formal statement.

Theorem 2.4. *For all integers $k \geq 1$, there are self-complementary graphs on $4k$ and $4k+1$ vertices.*

Proof. We induct on k. Our base cases for $k = 1$ are the path graph P_4 and the cycle graph C_5, which we've already seen.

Now assume that a self-complementary graph G on either $4k$ or $4k+1$ vertices exists; the same argument will actually apply to both cases. We'll use it to build a larger self-complementary graph with 4 more vertices (either $4(k+1)$ or $4(k+1)+1$ vertices).

Here's how: take G, and add 4 new vertices a_k, b_k, c_k, d_k. Between them, add edges a_kb_k, b_kc_k, and c_kd_k. Also, join every vertex in G by an edge to a_k and d_k (but *not* b_k or c_k).

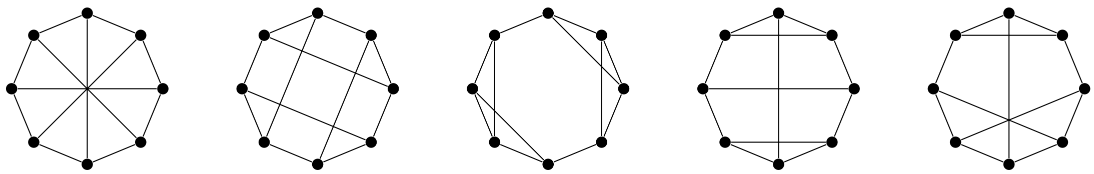


If $f\colon V(G) \to V(G)$ is an isomorphism from G to $\overline{G}$, extend it to the four new vertices by defining $f(a_k) = c_k$, $f(b_k) = a_k$, $f(c_k) = d_k$, and $f(d_k) = b_k$. You can check (looking at the diagram above) that this defines an isomorphism from the bigger graph to its complement, showing that it's also self-complementary.

By induction, we can generate self-complementary graphs of every possible size! □

3 Practice problems

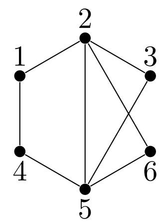
1. Prove that none of these five graphs are isomorphic: find invariants distinguishing them all from each other.



2. In fact, the five graphs shown above are the complete collection of 3-regular connected graphs on 8 vertices, up to isomorphism.

   In particular, this means that the cube graph Q_3 must be isomorphic to one of the five graphs above. Which one is it isomorphic to?

3. Find all automorphisms of the graph shown below: that is, all functions $f\colon \{1,2,3,4,5,6\} \to \{1,2,3,4,5,6\}$ that preserve the edges of the graph.



   (Hint: there are four, and one of them is very boring.)

4. Let G and H be isomorphic graphs. Prove the following:

   (a) G is connected if and only if H is connected.

   (b) G is bipartite if and only if H is bipartite.

   (c) G and H have the same number of vertices.

   (d) G and H have the same number of edges.

5. Suppose G and H are two graphs with the same vertex set: $V(G) = V(H)$. In the previous lecture, we talked about what happens if G and H have the same vertex degrees: for all vertices v, $\deg_G(v) = \deg_H(v)$.

   Prove that this neither implies nor is implied by G being isomorphic to H. In other words:

   (a) Give an example where $\deg_G(v) = \deg_H(v)$ for all vertices v, but G and H are not isomorphic.

   (b) Give an example where G and H are isomorphic, but $\deg_G(v)$ is not equal to $\deg_H(v)$ for at least some vertices v.

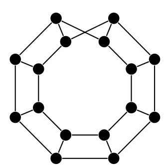
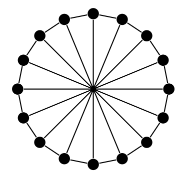
6. For $n \geq 5$, the walk (v_1, v_3, v_n) is a $v_1 - v_n$ walk in the complement $\overline{C}_n$.

   Use this fact and Lemma 2.1 to write a short proof that when $n \geq 5$, $\overline{C}_n$ is connected.

7. The “Möbius ladder” graph M_n is defined for every even $n > 4$, and it has two definitions:

   - According to one definition, we define it by starting with a path graph on vertices $v_1, v_2, \ldots, v_{n/2}$, another path graph on vertices $w_1, w_2, \ldots, w_{n/2}$, adding the edges $v_i w_i$ for $i = 1, 2, \ldots, n/2$, and finally adding the edges $v_1 w_{n/2}$ and $w_1 v_{n/2}$.
   - According to another definition, we define it by starting with a cycle graph on vertices $x_1, x_2, \ldots, x_n$, and adding an edge $x_i x_{i+n/2}$ for $i = 1, 2, \ldots, n/2$.

   The two diagrams below show the graph M_{16} according to each of the definitions.

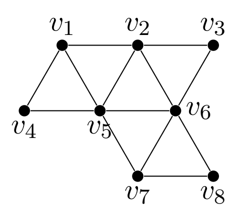
   
   

   Prove that for all even $n > 4$, the two definitions of M_n give isomorphic graphs.

8. Define two vertices v, w of a graph G to be **similar** if there is an automorphism of G that takes v to w.

   (a) Prove that if v and w are similar, then $G - v$ and $G - w$ are isomorphic.

   On the other hand, the converse to (a) is false! Let G be the graph below:

   (b) Prove that $G - v_4$ is isomorphic to $G - v_8$.

   (c) Prove that v_4 and v_8 are *not* similar. (In fact, G has no isomorphisms other than the identity automorphism.)