Math 482: Linear Programming	Mikhail Lavrov
Homework #2	
Spring 2020	*Due Friday, February 7*

1. Set up a linear program for the problem below. Do not solve. (Because there will be many similar constraints, it's fine if you just provide an example of each type of constraint, and say "do the same thing for every club" or "do the same thing for every pair of clubs".)

   Every student at a certain school is a member of at least one (maybe more) of its five clubs: Athletics Club, Book Club, Chess Club, Drama Club, and Ethics Club. Moreover, each club is very large, and includes at least $\frac{1}{2}$ of the students at the school.

   The school gives a "friendship award" to the two clubs that have the largest overlap in students. Every student that's a member of *both* clubs (not just one) will receive the award.

   What is the smallest possible fraction of students that can receive the friendship award?

   (Hint: to minimize the largest overlap between two clubs, minimize an auxiliary variable z that's greater than or equal to the size of every overlap. You will need to think carefully about what your other variables are in this linear program.)

2. The linear program

$$\begin{aligned} \underset{x,y\in\mathbb{R}}{\text{maximize}} \quad & 2x - 2y \\ \text{subject to} \quad & x - 3y \le 3 \\ & -4x + y \le 4 \\ & x - 2y \le 6 \\ & x, y \ge 0 \end{aligned}$$

   is unbounded. Use the simplex method to find a ray (a starting point (x_0, y_0) and a direction (u, v)) along which every point is a feasible solution, and the objective value increases arbitrarily far.

3. Use the two-phase simplex method to solve

$$\begin{aligned} \underset{x_1,x_2,x_3\in\mathbb{R}}{\text{minimize}} \quad & 3x_1 - x_2 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 5 \\ & 2x_1 + x_2 - 2x_3 \ge 6 \\ & x_1 + x_2 - x_3 \le 1 \\ & x_1, x_2, x_3 \ge 0. \end{aligned}$$

(There are more problems on the next page.)

4. Consider the following linear program:

$$\begin{aligned} \underset{x_1,x_2,x_3,x_4\in\mathbb{R}}{\text{maximize}} \quad & x_1 - 3x_2 - 2x_4 \\ \text{subject to} \quad & \frac{1}{2}x_1 - \frac{7}{2}x_2 - \frac{3}{2}x_3 + \frac{7}{2}x_4 \le 0 \\ & \frac{1}{2}x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 + \frac{1}{2}x_4 \le 0 \\ & x_1, x_2, x_3, x_4 \ge 0. \end{aligned}$$

(a) Perform two iterations of the simplex method using the following pivoting rule: choose the entering variable with the highest reduced cost. When both rows are valid leaving variables (in which case they'll always be tied for the smallest ratio: both ratios will always be 0) choose the basic variable for the first row as the leaving variable.

(b) Comparing the resulting tableau to the original tableau, argue that the simplex method with this pivoting rule will cycle forever, returning to the same tableau every six steps.

5. *(Only 4-credit students need to do this problem.)*

Consider a linear program of the form

$$\begin{aligned} \underset{\mathbf{x}\in\mathbb{R}^n}{\text{maximize}} \quad & \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \le \mathbf{b}. \end{aligned}$$

Suppose that points $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$ are both feasible solutions of this linear program.

(a) Show that if both $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions, then $\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$ is an optimal solution.

(b) Show that if $\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$ is an optimal solution, then both $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions.