Math 482: Linear Programming	Mikhail Lavrov
Homework #7	
Spring 2020	*Due Friday, April 3*

1. Show that any $n \times n$ matrix following the pattern

$$\begin{bmatrix} 1 & 0 & 1 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 1 \end{bmatrix}$$

is totally unimodular: any submatrix obtained by taking any k rows and any k columns has determinant -1, 0, or 1.

(It may help to consider the cases $k \leq 2$ and $k \geq 3$ separately.)

2. Consider the bipartite graph with vertices $\{a_1, a_2, \dots, a_{10}\}$ on one side, vertices $\{b_1, b_2, \dots, b_{10}\}$ on the other side, and an edge between a_i and b_j if the product ij is a multiple of 6.

Find a largest matching in this graph, and show that it cannot be any larger by finding a vertex cover of the same size.

3. A bipartite graph (X, Y, E) has $|X| = |Y| = n$ and is *r-regular*: every vertex (in X or in Y) is the endpoint of exactly r edges.

   (a) Determine $|E|$, the number of edges in the graph.

   (b) Show that any vertex cover must contain at least n vertices.

   (This implies that there is a matching of size n, which matches every vertex in X to a vertex in Y.)

4. Find examples of networks with the following properties:

   (a) A network with a unique maximum flow, but multiple minimum cuts.

   (b) A network with multiple maximum flows, but a unique minimum cut.

   (c) A network with multiple maximum flows and multiple minimum cuts.

   For each example, describe the maximum flow(s) and the minimum cut(s).

5. *(Only 4-credit students need to do this problem.)*

Consider a bipartite graph (X, Y, E) with $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_n\}$. The *biadjacency matrix* of this graph is the $m \times n$ matrix A where $A_{ij} = 1$ if there is an edge $(x_i, y_j) \in E$, and $A_{ij} = 0$ otherwise.

If $m = n$ (so that the matrix A is square) and $\det(A) = -3$, show that the graph contains a matching of size n.