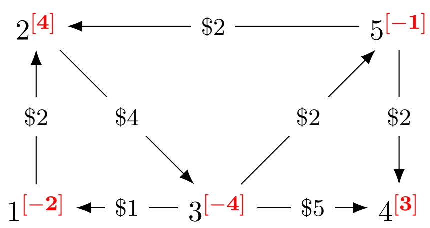
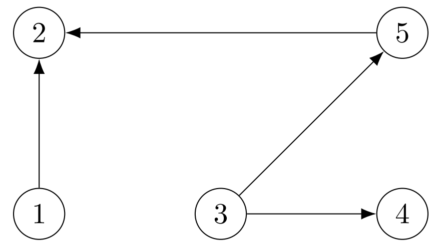
Math 482: Linear Programming	Mikhail Lavrov

Homework #9

Spring 2020	*Due Friday, April 24*

1. Consider the minimum-cost flow problem given in the first diagram below. The spanning tree shown in the second diagram gives a basic feasible solution.





   (a) Solve for the flows along each of the arcs in the basic feasible solution given by this spanning tree.

   (b) Determine the reduced cost of each of the nonbasic arcs in the network, then do a single pivoting step.

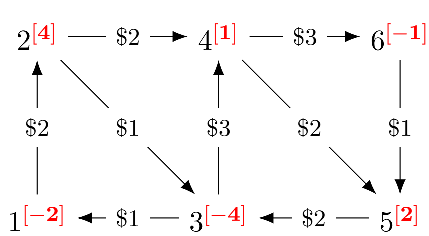
2. The dual of the min-cost flow problem is the following linear program:

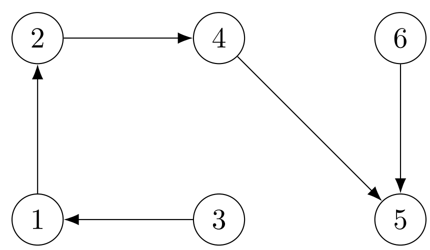
$$\begin{aligned} \underset{\mathbf{y}\in\mathbb{R}^{|N|}}{\text{maximize}} \quad & \sum_{k\in N} d_k y_k \\ \text{subject to} \quad & y_j - y_i \le c_{ij} \quad \text{for every } (i,j) \in A. \end{aligned}$$

   Given a spanning tree, we can find the corresponding dual solution using complementary slackness: arbitrarily set some $y_i = 0$ (due to the redundancy in the supply/demand constraints), then solve for $\mathbf{y}$ by using the equations $y_j - y_i = c_{ij}$ for every arc *in the tree.*

   (As with the primal solution, this dual solution is not feasible for all trees; usually, some of the constraints for arcs outside the tree will be violated.)

   (a) Find the dual basic solution corresponding to the spanning tree below (the network problem is on the left, and the tree is on the right).





   (b) Even when the dual basic solution is not feasible, it is still useful: the formula $c_{ij} + y_i - y_j$ gives the reduced cost of arc (i, j). For large networks, this is much faster than finding each reduced cost separately.

   Use this formula to solve for the reduced cost of the nonbasic arcs in this network.

3. Suppose that we are using the primal-dual method to solve the linear program

$$
\begin{aligned}
\underset{x_1,x_2,x_3,x_4\in\mathbb{R}}{\text{minimize}} \quad & 3x_1 + x_2 + x_4 \\
\text{subject to} \quad & x_1 + 2x_2 - 2x_3 - x_4 = 2 \\
& x_1 + x_2 \phantom{{}- 2x_3} - 2x_4 = 3 \\
& x_1 - 2x_2 - x_3 + 2x_4 = 4 \\
& x_1, x_2, x_3, x_4 \ge 0
\end{aligned}
$$

and that we are currently at the dual solution $\mathbf{u} = (1, 1, 1)$.

(a) Write down and solve (**RP**): the restricted primal program.

(b) Find the optimal solution $\mathbf{v}$ to (**DRP**): the dual of the restricted primal.

(c) Use $\mathbf{v}$ to find an improved dual solution to the original linear program.

(d) Show that the new dual solution is optimal by finding a corresponding optimal primal solution.

4. Consider the linear program

$$
\begin{aligned}
\underset{x_1,x_2,x_3,x_4\in\mathbb{R}}{\text{minimize}} \quad & x_1 + 3x_2 - 2x_3 - x_4 \\
\text{subject to} \quad & x_1 + 2x_2 - 2x_3 - x_4 = 2 \\
& x_1 + x_2 \phantom{{}- 2x_3} - 2x_4 = 3 \\
& x_1, x_2, x_3, x_4 \ge 0
\end{aligned}
$$

(a) Use the assumption that $x_1 + x_2 + x_3 + x_4 \le 100$ in all feasible solutions to the linear program above, to write down an equivalent linear program and a feasible dual solution for it.

(b) Perform two iterations of the primal-dual method, starting from the feasible dual solution you found.

5. *(Only 4-credit students need to do this problem.)*

Suppose that you have a network in which, instead of every arc having a capacity, there is a capacity associated through every node (other than the source s or the sink t). The flow along an arc can be arbitrary, but the total flow going into a node (equivalently, the total flow going out of a node) can be at most the capacity of that node.

Explain how to convert a maximum-flow problem for such a network into a standard maximum-flow problem.