Math 482: Linear Programming[1] Mikhail Lavrov

# Lecture 12: Linear Programming Duality

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## 1 An example of duality

Let's consider an example linear program:

$$\begin{aligned} \underset{x_1,x_2,x_3\in\mathbb{R}}{\text{maximize}} \quad & x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 + 4x_3 \le 3 \\ & x_1 + x_2 - 3x_3 \le 1 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

However, we're not going to try to solve this linear program. Instead, we want to prove some lower and upper bounds on the objective value of the solution.

Lower bounds for a maximization problem are easy.

- Setting $x_1 = x_2 = x_3 = 0$ satisfies both constraints, so clearly we can't do worse than an objective value of 0.
- We could try tweaking that: say $x_1 = 1$ and $x_2 = x_3 = 0$, then we get an objective value of 1.
- In general, any feasible solution gives us a lower bound on the objective value. If we wanted to get good lower bounds this way, we'd start trying to solve the linear program, which we said we didn't want to do.

What about upper bounds?

Well, here are some ideas:

- $x_1 + x_2$ is always less than or equal to $2x_1 + x_2 + 4x_3$. So if $2x_1 + x_2 + 4x_3 \le 3$, we can immediately conclude $x_1 + x_2 \le 3$.
- Note that we *can't* conclude from $x_1 + x_2 - 3x_3 \le 1$ that $x_1 + x_2 \le 1$, because the $-3x_3$ term could potentially make $x_1 + x_2 - 3x_3$ a lot smaller than $x_1 + x_2$.
- However, if we average the two constraints, we get an improvement:

$$\frac{1}{2}(2x_1 + x_2 + 4x_3) + \frac{1}{2}(x_1 + x_2 - 3x_3) \le \frac{1}{2}(3 + 1) \implies \frac{3}{2}x_1 + x_2 + \frac{1}{2}x_3 \le 2$$

  and we always have $x_1 + x_2 \le \frac{3}{2}x_1 + x_2 + \frac{1}{2}x_3$, so we conclude that $x_1 + x_2 \le 2$.

[1] This document comes from the Math 482 course webpage: https://faculty.math.illinois.edu/~mlavrov/courses/482-spring-2020.html

More generally, we could try to combine the two constraints with any coefficients. As long as $u_1 \geq 0$ and $u_2 \geq 0$, we can try to combine the inequalities with weights $u_1$ and $u_2$ to get

$$u_1(2x_1 + x_2 + 4x_3) + u_2(x_1 + x_2 - 3x_3) \leq 3u_1 + u_2.$$

This is a valid inequality, but not necessarily a useful one. We want the left-hand side to be an upper bound on $x_1 + x_2$. Rearranging the inequality we get as

$$(2u_1 + u_2)x_1 + (u_1 + u_2)x_2 + (4u_1 - 3u_2)x_3 \leq 3u_1 + u_2.$$

We would like the first two coefficients, $2u_1 + u_2$ and $u_1 + u_2$, to both be at least 1, because then the first two terms are at least $x_1 + x_2$. We would like the third coefficient, $4u_1 - 3u_2$, to be nonnegative. So we can write down another linear program answering the question: "what is the best upper bound we can find for the original linear program by combining the inequalities in this way?" That linear program is

$$\begin{aligned}
\underset{u_1, u_2 \in \mathbb{R}}{\text{minimize}} \quad & 3u_1 + \phantom{3}u_2 \\
\text{subject to} \quad & 2u_1 + \phantom{3}u_2 \geq 1 \\
& \phantom{2}u_1 + \phantom{3}u_2 \geq 1 \\
& 4u_1 - 3u_2 \geq 0 \\
& u_1, u_2 \geq 0
\end{aligned}$$

## 2 Weak and strong duality

We can do this for any linear program. Say we have the linear program

$$(\mathbf{P}) \qquad \begin{cases} \underset{\mathbf{x} \in \mathbb{R}^n}{\text{maximize}} & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{cases}$$

where $A$ is an $m \times n$ matrix. Then for any coefficients $\mathbf{u} \in \mathbb{R}^m$ with $\mathbf{u} \geq \mathbf{0}$, we can deduce the inequality $\mathbf{u}^\mathsf{T}A\mathbf{x} \leq \mathbf{u}^\mathsf{T}\mathbf{b}$, which corresponds to multiplying the $i^{\text{th}}$ inequality by $u_i$ and then adding them all together.

The row vector of coefficients in the resulting inequality is $\mathbf{u}^\mathsf{T}A$, and in order to deduce a bound on $\mathbf{c}^\mathsf{T}\mathbf{x}$, we would like to have $\mathbf{u}^\mathsf{T}A \geq \mathbf{c}^\mathsf{T}$. The upper bound we get is $\mathbf{u}^\mathsf{T}\mathbf{b}$, and we want this to be as small as possible. This gives us a new linear program called the *dual linear program*:

$$(\mathbf{D}) \qquad \begin{cases} \underset{\mathbf{u} \in \mathbb{R}^m}{\text{minimize}} & \mathbf{u}^\mathsf{T}\mathbf{b} \\ \text{subject to} & \mathbf{u}^\mathsf{T}A \geq \mathbf{c}^\mathsf{T} \\ & \mathbf{u} \geq \mathbf{0} \end{cases} \iff \begin{matrix} \underset{\mathbf{u} \in \mathbb{R}^m}{\text{minimize}} & \mathbf{b}^\mathsf{T}\mathbf{u} \\ \text{subject to} & A^\mathsf{T}\mathbf{u} \geq \mathbf{c} \\ & \mathbf{u} \geq \mathbf{0} \end{matrix}$$

(When we have the dual linear program around, we call the linear program we started with the *primal linear program*.)

In the formulation of the dual linear program on the right, we also took the transpose of both sides, putting into a form more usual for linear programs. But it's also convenient to use the formulation on the left, because then the dual program is distinguished by being in terms of a row vector $\mathbf{u}^\mathsf{T}$ instead of a column vector $\mathbf{u}$.

Just due to our reasoning for how we constructed the dual linear program, we know that there is a relationship between the primal and the dual. This relationship is called *weak duality*:

**Theorem 2.1** (Weak duality of linear programs)**.** *For any* $\mathbf{x} \in \mathbb{R}^n$ *which is feasible for the primal linear program* ($\mathbf{P}$) *(or* primal feasible*) and for any* $\mathbf{u} \in \mathbb{R}^m$ *which is feasible for the dual linear program* ($\mathbf{D}$) *(or* dual feasible*), we have* $\mathbf{c}^\mathsf{T}\mathbf{x} \le \mathbf{b}^\mathsf{T}\mathbf{u}$*.*

*In particular, the objective value of the dual optimal solution is an upper bound for the objective value of the primal optimal solution (assuming both optimal solutions exist).*

There is a further fact called *strong duality*. It is by no means obvious; maybe it is surprising. We have not proved it yet, but we will eventually. Strong duality says:

**Theorem 2.2** (Strong duality of linear programs)**.** *If either one of* ($\mathbf{P}$) *or* ($\mathbf{D}$) *has an optimal solution, then so does the other one. The objective values of the optimal solutions are equal.*

In other words, the dual program is *good* at finding bounds on the primal program: the best bound it finds is exactly correct.

# 3 Duals of other kinds of programs

So far we've discussed starting with a primal program that's a maximization problem with nonnegative variables and an $A\mathbf{x} \le \mathbf{b}$ constraint. Duality is more general than this: it can handle any kind of linear program. The only thing that never changes is that

**Variables in one program correspond to constraints in the other.**

Just to give a few examples of how things change:

- Suppose that we drop the requirement that $x_2 \ge 0$ in our original linear program.

  In that case, for an inequality we generate to give an upper bound on $x_1 + x_2$, the coefficient of $x_2$ has to be *exactly* 1. As before, if we have something like $2x_1 + \frac{1}{2}x_2 + 3x_3$, that's not an upper bound on $x_1 + x_2$, because $\frac{1}{2}x_2$ might be less than $x_2$, which is a problem if $x_1, x_3$ are small and $x_2$ is large. However, $2x_1 + 2x_2 + 3x_3$ is also not an upper bound on $x_1 + x_2$ anymore, because $x_2$ could be negative, in which case $2x_2 < x_2$.

  So an unconstrained variable gives us $=$ constraints in the dual linear program.

- Suppose that we reverse the first constraint to say $2x_1 + x_2 + 4x_3 \ge 3$.

  In that case, to get an upper bound, we have to multiply this constraint by something negative, so that the inequality is reversed.

  So a $\ge$ constraint gives us a nonpositive variable in the dual linear program.

- Suppose that the primal program asks to *minimize* $x_2 + x_2$ instead of maximizing it.

This messes with everything, because now we are trying to get lower bounds instead of upper bounds. In particular, the relationship between (**P**) and (**D**) is reversed: a feasible solution for (**P**) will always have a greater or equal objective value compared to a feasible solution for (**D**).

Now $\leq$ constraints in (**P**) correspond to nonnegative variables in (**D**) (they are the "natural" kind of constraint when we're minimizing) and $\geq$ constraints in (**P**) correspond to nonpositive variables in (**D**).

Actually, the relationship between (**P**) and (**D**) is symmetric: if (**D**) is the dual of (**P**), then (**P**) is the dual of (**D**). It's easiest to describe the duality relationship as a relationship between a maximization problem and a minimization problem, never mind which one of them was the primal and which was the dual.

With that in mind, here is the complete list of possible correspondences betwene a constraint in one problem and a variable in the other:

| Maximization problem | Minimization problem |
|---|---|
| $\leq$ constraint | variable $\geq 0$ |
| $=$ constraint | unconstrained variable |
| $\geq$ constraint | variable $\leq 0$ |
| variable $\geq 0$ | $\geq$ constraint |
| unconstrained variable | $=$ constraint |
| variable $\leq 0$ | $\leq$ constraint |

Memorizing the rules in the table is possible, but it probably isn't very satisfying. It is healthier to practice figuring out the correspondence for yourself, by asking the questions in the examples above: how can we combine the constraints of the primal problem to get bounds on its optimal value, of whichever kind makes sense?