Math 482: Linear Programming[1] Mikhail Lavrov

Lecture 18: Zero-Sum Games I

March 6, 2020 *University of Illinois at Urbana-Champaign*

1 Introduction to games

1.1 Matrix games

We are going to consider two-player games where the players simultaneously pick strategies, and are rewarded with a *payoff* based on their choices. A classic example is the prisoner's dilemma (which we're only going to briefly mention).

Here, the two players (Alice and Bob) are suspected of a bank robbery (2 years in prison), but arrested only for some minor thing like tax evasion (1 year in prison). The prosecutor offers each of them a bargain: testify against the other, and the tax evasion charges are dropped. As a result:

- If Alice and Bob both stay silent, each spends 1 year in prison.
- If Alice testifies against Bob, Alice goes free and Bob spends 3 years in prison (and vice versa).
- If Alice and Bob both testify against each other, they both spend 2 years in prison.

We can represent spending t years in prison by a payoff of $-t$. (The goal is always to maximize your payoff.)

We can represent the prisoner's dilemma (and more generally, any such game) by a payoff matrix. Assign each of Alice's strategies a row, and each of Bob's strategies a column. At the intersection of the row and the column, record Alice and Bob's respective payoffs:

	Bob stays silent	Bob testifies
Alice stays silent	$(-1, -1)$	$(-3, 0)$
Alice testifies	$(0, -3)$	$(-2, -2)$

It is best for both players to stay silent, rather than both testify, so they should agree not to testify. However, each individual player is better of testifying, no matter what the other does, so they should betray that agreement. However, that leaves both players worse off. This weird behavior gives the prisoner's dilemma a rich and complicated dynamic...

1.2 Zero-sum games

...which we're going to ignore entirely, because in this class we will only consider zero-sum games.

[1] This document comes from the Math 482 course webpage: https://faculty.math.illinois.edu/~mlavrov/courses/482-spring-2020.html

A zero-sum game is one in which any hope of cooperation between the players is eliminated, because Alice's payoff is the negative of Bob's payoff. (They sum to 0.) Whatever outcome helps one player, hurts the other player equally.

In other words, we can represent each outcome by a single number. (Let's say it's Alice's payoff.) Alice's objective in the game is to maximize that number. Bob's objective in the game is to minimize it. (We can think of this number as a dollar amount that Bob gives to Alice: maybe it's negative, in which case actually it's Alice giving money to Bob.)

Our goal in analyzing these games will be to determine what Alice and Bob's optimal strategies are, and what the resulting payoff is.

2 Strategies

We will look at some examples of zero-sum games to illustrate a few cases where we can find the optimal strategies easily, and the general case which is more complicated.

2.1 Dominated strategies

Consider the following game, called "higher number". It's a pretty stupid game. Alice and Bob each hold up one, two, or three fingers. The player holding up fewer fingers gives $1 to the other player. The payoff matrix (representing Alice's payoffs) is:

	Bob: one	Bob: two	Bob: three
Alice: one	0	-1	-1
Alice: two	1	0	-1
Alice: three	1	1	0

It is immediate to see that both players want to hold up more fingers rather than fewer. There's two notions that make this formal:

- No matter what Alice does, Bob is better off holding up up two fingers than one.

  In general, we say that one strategy of a player *dominates* another strategy if, no matter what the other player does, the first strategy is better than (or at least as good as) the second.

  Here, "two" dominates "one" and "three" dominates both "one" and "two", for both players.

- No matter what Alice does, Bob is best off holding up three fingers.

  In general, if a strategy dominates every other strategy, we call it a *dominant strategy.*

Games with a dominant strategy are really easy to analyze. If a player has a dominant strategy, that's the optimal strategy, and we can assume that they will play it. With that assumption, the other player should just pick the best response to that strategy.

Even if there is no dominant strategy, it makes sense to eliminate from consideration any strategy that's dominated by another strategy. After all, that other strategy is never worse. This can help us simplify the problem and make the payoff matrix smaller.

2.2 Saddle points

Consider the following game, which I don't have a nice description of. (As before—and as always from now on—the entries in the matrix are the row player Alice's payoffs.)

	B_1	B_2	B_3
A_1	0	-1	3
A_2	1	3	2
A_3	-1	4	-2

I have carefully cooked up a table in which there is no dominant strategy. Nevertheless, the game becomes easy to analyze after we make two observations:

- If Alice chooses strategy A_2, she guarantees herself a payoff of at least \$1. Depending on what Bob does, of course, she could get even more.
- If Bob chooses strategy B_1, he guarantees that he will have to give no more than \$1 to Alice. Depending on what Alice does, of course, he could do even better or even come out ahead.

This means that for Alice, following strategy A_2 is optimal. It guarantees her \$1, and we know she can't get more than that, because Bob has a strategy limiting his losses to \$1. Similarly, for Bob, following strategy B_1 is optimal, because it limits his losses to \$1, and we know he can't dod better than that, because Alice has a strategy to win at least that much.

(Either one of them can even announce the strategy they're following before the game: it won't make a difference!)

In general, we call an outcome like (A_2, B_1) in this case a *saddle point*. A saddle point is an outcome which is the minimum of its row, but the maximum of its column. Whenever there is a saddle point, either player can guarantee an outcome at least as good as the saddle point by choosing its row or its column as a strategy.

2.3 Mixed strategies

Finally, we will consider the *even-odd game*. In this game, Alice and Bob each hold up either 1 or 2 fingers. Let N be the total. Alice wants an odd total: if N is odd, Bob gives Alice N dollars. Bob wants an even total: if N is even, Alice gives Bob N dollars. The payoff matrix is

	Bob: 1	Bob: 2
Alice: 1	-2	3
Alice: 2	3	-4

This game has neither a dominant strategy for either player, nor a saddle point. What can we do?

If Bob knew what Alice would play, he could play the same move in response and win either \$2 or \$4. So there's no best move for Alice.

One possible strategy for Alice is to flip a coin before making her move (invisibly from Bob). If it's heads, Alice holds up 1 finger; if it's tails, Alice holds up 2 fingers. If Alice is doing this, we can compute Bob's *expected payoff* from each of his options:

	Bob: 1	Bob: 2
Alice: coin flip	$\frac{-2+3}{2} = 0.50$	$\frac{3+-4}{2} = -0.50$

The coin-flip strategy has better worst-case behavior for Alice: even if Bob knows that Alice is planning to flip a coin, Bob can at best guarantee himself 50 cents (on average) by holding up 2 fingers rather than 1.

In general, Alice has many different *mixed strategies* in which she chooses a strategy randomly. If Alice has a different options, we can represent a mixed strategy by a vector of probabilities: $\mathbf{x} \in \mathbb{R}^a$ such that $\mathbf{x} \geq 0$ and $x_1 + x_2 + \cdots + x_a = 1$. If A is the $a \times b$ payoff matrix for Alice, then $\mathbf{x}^\mathsf{T} A$ is the row vector of Alice's expected payoffs, in the case that Alice plays the mixed strategy $\mathbf{x}$, for each of Bob's choices.

Bob can also play a mixed strategy. If Bob has b different options, then we can represent a mixed strategy for Bob by a vector $\mathbf{y} \in \mathbb{R}^b$ such that $\mathbf{y} \geq 0$ and $y_1 + y_2 + \cdots + y_b = 1$. If Bob plays this mixed strategy, Alice's expected payoffs are given by the column vector $A\mathbf{y}$.

Finally, suppose that both players are playing mixed strategies: say, Alice holds up 1 or 2 fingers with probabilities x_1, x_2 respectively while Bob holds up 1 or 2 fingers with probabilities y_1, y_2 respectively. Then Alice's expected payoff is the sum of

$$\text{(probability of outcome)} \times \text{(payoff from outcome)}$$

over all outcomes in the table. In this case, this is

$$x_1 y_1 \cdot (-2) + x_1 y_2 \cdot (3) + x_2 y_1 \cdot (3) + x_2 y_2 \cdot (-4).$$

Both in this case and in the general, the expected payoff for Alice is given by

$$\mathbf{x}^\mathsf{T} A \mathbf{y}.$$