

## Lecture 3: The Simplex Method

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## 0 The plan

The simplex method can be roughly summarized as “go from one solution to another, improving every time, until you reach the best solution”. We’ll get there in two steps.

Today, we will talk about how we go from one solution to another. We will only think about the constraints of our linear program, and not even consider the objective function.

In the next lecture, we’ll go back and think about which steps bring us closer to our goal, and which steps take us further away from it.

## 1 A quick review of linear algebra

A good chunk of your typical linear algebra class consists of solving systems of linear equations; for example,

$$\begin{cases} 3x + y = 6 \\ x - y = -2 \end{cases}$$

We can solve this problem by putting the entries into a matrix and row-reducing. (I mean, for a problem this size, we probably shouldn’t bother, but whatever.)

$$\left[ \begin{array}{cc|c} 3 & 1 & 6 \\ 1 & -1 & -2 \end{array} \right] \rightsquigarrow \left[ \begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \end{array} \right]$$

Read as equations again, the rows of the resulting matrix become “ $x = 1$ ” and “ $y = 3$ ”, telling us the solution.

Things become more complicated if we have more variables than equations; in this case, the typical thing to expect is a family of infinitely many solutions. Given the system

$$\begin{cases} 3x + y + 5z = 6 \\ x - y + 3z = -2 \end{cases}$$

we can row-reduce again, and the result looks like this:

$$\left[ \begin{array}{ccc|c} 3 & 1 & 5 & 6 \\ 1 & -1 & 3 & -2 \end{array} \right] \rightsquigarrow \left[ \begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 3 \end{array} \right]$$

The  $x$  and  $y$  columns look “solved” but the  $z$  column looks like a bunch of random numbers, which is typical.

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<sup>1</sup>This document comes from the Math 482 course webpage: <https://faculty.math.illinois.edu/~mlavrov/courses/482-spring-2020.html>

We call  $x$  and  $y$  in this case the *basic variables* and  $z$  and any others like it the *nonbasic variables*.

We can get one solution to the system of equations just by setting all nonbasic variables to 0. In this case, we can ignore the  $z$  column, and just get  $x = 1$  and  $y = 3$  as before. This is called the *basic solution*. It is the nicest to describe in an infinite family of solutions: we could set  $z$  to any value, and it would be almost as easy to find  $x$  and  $y$ .

In general, we can set the nonbasic variables to anything we like, and get a solution by finding what the basic variables must be. But the nicest solution is the basic solution, which comes from setting the nonbasic variables to 0.

Linear algebra classes don't make a point of it, but choosing the first variables we can to be the basic variables is just a convention. Instead, we could have made  $y$  and  $z$  the basic variables, row-reducing to get

$$\left[ \begin{array}{ccc|c} 3 & 1 & 5 & 6 \\ 1 & -1 & 3 & -2 \end{array} \right] \rightsquigarrow \left[ \begin{array}{ccc|c} 1/2 & 1 & 0 & 7/2 \\ 1/2 & 0 & 1 & 1/2 \end{array} \right]$$

This works just as well, but now our basic solution sets  $y = \frac{7}{2}$  and  $z = \frac{1}{2}$  (and the nonbasic variable  $x$  to 0), and we get infinitely many other solutions by varying  $x$ .

## 2 Back to linear programming

The simplex method works on linear programs in equational form: the constraints are  $A\mathbf{x} = \mathbf{b}$  with  $\mathbf{x} \geq \mathbf{0}$ . That is, we have a perfectly ordinary system of linear equations, together with the added constraint that all variables must be nonnegative.

There are still infinitely many feasible solutions, but on the first day, we saw a rule that cuts their number down to a manageable amount:

**Rule #1: At least one optimal solution is a vertex of the feasible region.<sup>2</sup>**

The vertices of the feasible region are the points where we meet the boundaries of as many inequalities as possible. If our only inequalities are " $x_1, x_2, \dots, x_n \geq 0$ ", then the vertices are the points where as many variables as possible are 0. If we're solving the system of equations the way we did earlier, we'd like to set all the nonbasic variables to achieve this. This tells us another rule:

**Rule #2: All vertices of the feasible region are basic solutions of the system of linear equations.**

This gives a motivation to find as many basic solutions as possible.

Consider the following example: a linear program with constraints

$$\begin{cases} x_1 & + x_3 + 3x_4 + x_5 = 4 \\ & x_2 & + x_4 + x_5 = 3 \\ & & x_3 + x_4 - x_5 = 1 \\ x_1, x_2, x_3, x_4, x_5 & \geq 0. \end{cases}$$

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<sup>2</sup>Terms and conditions apply. Void if the linear program doesn't have an optimal solution. Also void if the feasible region doesn't have any vertices.

Here,

$$A = \begin{bmatrix} 1 & 0 & 1 & 3 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}.$$

It's easy to start with basic variables  $x_1, x_2, x_3$ : subtract the third row from the first. We get

$$\begin{cases} x_1 & + 2x_4 + 2x_5 = 3 \\ & x_2 & + x_4 + x_5 = 3 \\ & & x_3 + x_4 - x_5 = 1 \\ x_1, x_2, x_3, x_4, x_5 \geq 0. \end{cases}$$

and if we set  $x_4 = x_5 = 0$ , we can read off the basic variables:  $x_1 = 3$ ,  $x_2 = 3$ , and  $x_3 = 1$ .

There is special notation for this step. Let  $\mathcal{B} = (1, 2, 3)$ : the indices of the basic variable positions. Write

$$A_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for the submatrix of  $A$  where we take only the columns from  $\mathcal{B}$ . Then the row reduction we did can be summarized by the matrix multiplication  $A_{\mathcal{B}}^{-1}A\mathbf{x} = A_{\mathcal{B}}^{-1}\mathbf{b}$ , or

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 & 3 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

which simplifies, as expected, to

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}.$$

The values of the basic variables are given by  $\mathbf{x}_{\mathcal{B}} = A_{\mathcal{B}}^{-1}\mathbf{b}$ .

Let's try this again with a different basis:  $\mathcal{B}' = (2, 3, 4)$ . Here,

$$A_{\mathcal{B}'} = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad A_{\mathcal{B}'}^{-1} = \begin{bmatrix} -1/2 & 1 & 1/2 \\ -1/2 & 0 & 3/2 \\ 1/2 & 0 & -1/2 \end{bmatrix}$$

and the new system of equations  $A_{\mathcal{B}'}^{-1}A\mathbf{x} = A_{\mathcal{B}'}^{-1}\mathbf{b}$  simplifies to

$$\begin{bmatrix} -1/2 & 1 & 0 & 0 & 0 \\ -1/2 & 0 & 1 & 0 & -2 \\ 1/2 & 0 & 0 & 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3/2 \\ -1/2 \\ 3/2 \end{bmatrix}.$$

The values of the basic variables are  $(x_2, x_3, x_4) = \mathbf{x}_{\mathcal{B}'} = A_{\mathcal{B}'}^{-1}\mathbf{b} = (\frac{3}{2}, -\frac{1}{2}, \frac{3}{2})$ .

There's a problem. With  $\mathcal{B} = (1, 2, 3)$ , we got  $(x_1, x_2, x_3, x_4, x_5) = (3, 3, 1, 0, 0)$ , which does satisfy  $\mathbf{x} \geq 0$ . With  $\mathcal{B}' = (2, 3, 4)$ , we got  $(x_1, x_2, x_3, x_4, x_5) = (0, \frac{3}{2}, -\frac{1}{2}, \frac{3}{2}, 0)$ , which doesn't satisfy it.

We don't just want a basic solution. We want a *basic feasible solution*: a basic solution which also satisfies the nonnegativity constraints.

The simplex method ensures this by a strategy called *pivoting*. The idea is that

1. We start with a basic feasible solution.
2. We modify it slightly by making one nonbasic variable enter the basis, and one basic variable leave the basis.
3. We choose which variable leaves to avoid negative signs, so that we arrive at a new basic feasible solution.

Since we need to start with a basic feasible solution, let's return to  $\mathcal{B} = (1, 2, 3)$ , with basic solution  $\mathbf{x} = (3, 3, 1, 0, 0)$ . For no particular reason, let's pick  $x_5$  to enter the basis. Then  $x_4$  remains 0, so we can ignore the  $x_4$  column, but as  $x_5$  changes, the other variables change in terms of  $x_2$ :

$$\begin{cases} x_1 & + 2x_5 = 3 \\ & x_2 + x_5 = 3 \\ & x_3 - x_5 = 1 \end{cases} \implies \begin{cases} x_1 = 3 - 2x_5 \\ x_2 = 3 - x_5 \\ x_3 = 1 + x_5 \end{cases}$$

As we increase  $x_5$ , the other variables will decrease. Once  $x_5$  reaches  $\frac{3}{2}$ ,  $x_1$  will drop to 0. We can't increase  $x_5$  any further, or  $x_1$  will become negative, and our point will no longer be feasible.

Since  $x_1$  is the variable that drops to 0 first, it's the one that leaves the basis. Our new basis is  $\mathcal{B}'' = (2, 3, 5)$ .

In principle, we could recompute the entire matrix by finding  $A_{\mathcal{B}''}^{-1}$  and simplifying  $A_{\mathcal{B}''}^{-1}A\mathbf{x} = A_{\mathcal{B}''}^{-1}\mathbf{b}$ . That's too much work, though. Instead, we can proceed by row reduction.

Since  $x_1$  (the basic variable with a pivot in the first row) leaves, we want to make  $x_5$  the new basic variable with a pivot in the first row. So the boxed entry should be the new pivot, and row reduction leads us to the next matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 2 & \boxed{2} \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1/2 & 0 & 0 & 1 & 1 \\ -1/2 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1 & 2 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3/2 \\ 3/2 \\ 5/2 \end{bmatrix}$$

How can we quickly tell which variable leaves the basis? Two factors go into figuring out which variable drops to 0 first:

- The rate at which they change as we increase  $x_5$ , given by  $x_5$ 's column  $(2, 1, -1)$ .
- The starting values of the variables, given by the right-hand side  $(3, 3, 1)$ .

A "first pass" through the variables tells us that  $x_3$  is never going to hit 0: the  $-1$  in  $x_5$ 's column tells us that the rate of change is  $+1$ . So we can forget about  $x_3$ , and compare only  $x_1$  and  $x_2$ .

To compare them, we can take the ratios  $\frac{\text{starting value}}{\text{rate of decrease}}$ , which is  $\frac{3}{2}$  for  $x_1$  and  $\frac{3}{1} = 3$  for  $x_2$ . Since  $x_1$  has the smaller value, it's the first to drop to 0, so it's the one that leaves the basis.

In general, the strategy is to compare all rows with a positive entry in the entering variable's column. Then, take the ratio  $\frac{\text{rightmost column}}{\text{entering variable's column}}$  for each such row, and pick the variable with the smallest ratio to leave the basis.