Math 482: Linear Programming[1] Mikhail Lavrov

Lecture 8: Vertices, Extreme Points, and Basic Feasible Solutions

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1 Basic feasible solutions

Let's suppose we are solving a general linear program in equational form:

$$\begin{aligned} \underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \quad & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

Here, A is an $m \times n$ matrix, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{c} \in \mathbb{R}^n$. Today, we will assume that the rows of A are linearly independent. (If not, then either the system $A\mathbf{x} = \mathbf{b}$ has no solutions, or else some of the equations are redundant. In the first case, we just forget about analyzing such a linear program; in the second case, we can begin by deleting the redundant rows.)

We've informally said that a *basic feasible solution* is one in which "as many of the variables as possible" are 0. This is not quite precise: in some cases (due to degeneracy) it's possible to have unusually many 0 values, and we don't want this to mess with our definition. Instead we make the definition as follows.

Choose some m-tuple of columns (or of variables) $\mathcal{B}$ to be basic. We want $\mathcal{B}$ to be ordered, because our tableaux look slightly different when the basic variables are chosen in a different order. For convenience, we let $\mathcal{N}$ be the $(n-m)$-tuple of nonbasic variables: those that are not in $\mathcal{B}$.

We can split up vectors and matrices into the basic and nonbasic part. For example, if $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$, $\mathcal{B} = (2,4)$, and $\mathcal{N} = (1,3,5)$, we have $\mathbf{x}_\mathcal{B} = (x_2, x_4)$ and $\mathbf{x}_\mathcal{N} = (x_1, x_3, x_5)$. This can also be done with A and $\mathbf{c}$: we can write the objective function as

$$\mathbf{c}^\mathsf{T}\mathbf{x} = \mathbf{c}_\mathcal{B}{}^\mathsf{T}\mathbf{x}_\mathcal{B} + \mathbf{c}_\mathcal{N}{}^\mathsf{T}\mathbf{x}_\mathcal{N}$$

and the system of equations $A\mathbf{x} = \mathbf{b}$ as

$$A_\mathcal{B}\mathbf{x}_\mathcal{B} + A_\mathcal{N}\mathbf{x}_\mathcal{N} = \mathbf{b}.$$

To get a basic solution, we want to choose $\mathcal{B}$ so that $A_\mathcal{B}$ (an $m \times m$ matrix) is invertible. This is always possible if the rows of A are linearly independent. Not every choice of $\mathcal{B}$ will work: for example, in 2 dimensions, if two of the sides of the feasible region are parallel lines, they never intersect.

Now set $\mathbf{x}_\mathcal{N} = \mathbf{0}$, and $\mathbf{x}_\mathcal{B} = A_\mathcal{B}^{-1}\mathbf{b}$. This satisfies $A\mathbf{x} = \mathbf{b}$. If, additionally, we have $\mathbf{x}_\mathcal{B} \geq \mathbf{0}$ (we always have $\mathbf{x}_\mathcal{N} \geq \mathbf{0}$, because $\mathbf{x}_\mathcal{N} = \mathbf{0}$), we call $\mathbf{x}$ a *basic feasible solution.*

[1] This document comes from the Math 482 course webpage: https://faculty.math.illinois.edu/~mlavrov/courses/482-spring-2020.html

2 Other notions of corner points

There are other notions of "corner point" besides a basic feasible solution. We say that

- A *vertex* of a set $S \subseteq \mathbb{R}^n$ is a point $\mathbf{x} \in S$ such that some linear function $\boldsymbol{\alpha}^\mathsf{T}\mathbf{x}$ is strictly minimized at $\mathbf{x}$: $\boldsymbol{\alpha}^\mathsf{T}\mathbf{x} < \boldsymbol{\alpha}^\mathsf{T}\mathbf{y}$ for any $\mathbf{y} \in S$, $\mathbf{y} \neq \mathbf{x}$.
- An *extreme point* of a set $S \subseteq \mathbb{R}^n$ is a point $\mathbf{x} \in S$ that does not lie between any other points of S. Formally, if $\mathbf{x}$ is an extreme point if, whenever $\mathbf{x} \in [\mathbf{y}, \mathbf{y}']$ for $\mathbf{y}, \mathbf{y}' \in S$, either $\mathbf{x} = \mathbf{y}$ or $\mathbf{x} = \mathbf{y}'$.

  In other words, if $\mathbf{x}$ can be written as $t\mathbf{y} + (1-t)\mathbf{y}'$ for $\mathbf{y}, \mathbf{y}' \in S$ and $0 \le t \le 1$, either $\mathbf{x} = \mathbf{y}$ (and we can set $t = 1$) or $\mathbf{x} = \mathbf{y}'$ (and we can set $t = 0$).

When the set we're considering is $F = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}, \mathbf{x} \ge \mathbf{0}\}$, the feasible region of a linear program, all three notions—basic feasible solution, vertex, and extreme point—are the same. This is what we'll try to prove today.

2.1 From basic feasible solutions to vertices

Proposition 2.1. *Any basic feasible solution is a vertex of the feasible region.*

Proof. Take any choice of basic and nonbasic variables $(\mathcal{B}, \mathcal{N})$ for which setting $\mathbf{x}_\mathcal{N} = \mathbf{0}$ produces a basic feasible solution. Define $\boldsymbol{\alpha}$ by

$$\alpha_i = \begin{cases} 1 & i \in \mathcal{N}, \\ 0 & i \in \mathcal{B}. \end{cases}$$

Then $\boldsymbol{a}^\mathsf{T}\mathbf{x}$ is the sum of the nonbasic variables in $\mathbf{x}$.

Since $\mathbf{x}_\mathcal{N} \ge \mathbf{0}$, $\boldsymbol{a}^\mathsf{T}\mathbf{x}$ is minimized exactly when we set $\mathbf{x}_\mathcal{N} = \mathbf{0}$. And that's exactly the basic feasible solution corresponding to $(\mathcal{B}, \mathcal{N})$. □

2.2 From vertices to extreme points

Proposition 2.2. *Any vertex of a set $S \subseteq \mathbb{R}^n$ is also an extreme point of S. (In particular, any basic feasible solution is also an extreme point of the feasible region.)*

Proof. Let $\mathbf{x} \in S$ be a vertex of S, and et $\boldsymbol{\alpha}$ be the vector such that $\boldsymbol{\alpha}^\mathsf{T}\mathbf{x} < \boldsymbol{\alpha}^\mathsf{T}\mathbf{y}$ for any $\mathbf{y} \in S$ with $\mathbf{y} \neq \mathbf{x}$.

Suppose, for the sake of contradiction, that $\mathbf{x}$ lies on the line segment $[\mathbf{y}, \mathbf{y}']$ with $\mathbf{y}, \mathbf{y}' \in S$ and $\mathbf{y}, \mathbf{y}' \neq \mathbf{x}$. This is what it means to *not* be an extreme point.

The $\mathbf{x} = t\mathbf{y} + (1-t)\mathbf{y}'$, so

$$\boldsymbol{\alpha}^\mathsf{T}\mathbf{x} = t(\boldsymbol{\alpha}^\mathsf{T}\mathbf{y}) + (1-t)(\boldsymbol{\alpha}^\mathsf{T}\mathbf{y}') > t(\boldsymbol{\alpha}^\mathsf{T}\mathbf{x}) + (1-t)(\boldsymbol{\alpha}^\mathsf{T}\mathbf{x}) = \boldsymbol{\alpha}^\mathsf{T}\mathbf{x},$$

using the inequalities $\boldsymbol{\alpha}^\mathsf{T}\mathbf{y} > \boldsymbol{\alpha}^\mathsf{T}\mathbf{x}$ and $\boldsymbol{\alpha}^\mathsf{T}\mathbf{y}' > \boldsymbol{\alpha}^\mathsf{T}\mathbf{x}$, and we get a contradiction.

Therefore $\mathbf{x}$ must be an extreme point. □

2.3 From extreme points to basic feasible solutions

The last step, going from extreme points to basic feasible solutions, is trickier.

Proposition 2.3. *Any extreme point of the feasible region is a basic feasible solution.*

Proof. Let $\mathbf{x}$ be any extreme point of the feasible region. Define $\mathcal{W} = \{i : x_i \neq 0\}$ and $\mathcal{Z} = \{i : x_i = 0\}$, by analogy with $\mathcal{B}$ and $\mathcal{N}$ for a basic feasible solution.

We can't really ask whether $A_\mathcal{W}$ is invertible or not, because we have no reason to even think it's square. But what we can do is ask whether we can find a nonzero $\mathbf{u} \in \mathbb{R}^n$ such that:

$$\begin{cases} A_\mathcal{W}\mathbf{u}_\mathcal{W} = \mathbf{0} \\ \mathbf{u}_\mathcal{Z} = \mathbf{0}. \end{cases}$$

If there is no such $\mathbf{u}$, then we'll argue that $\mathbf{x}$ is a basic feasible solution. If there is such an $\mathbf{u}$, then we'll argue that $\mathbf{x}$ actually wasn't an extreme point, and get a contradiction.

First, suppose there is no such $\mathbf{u}$: whenever $A_\mathcal{W}\mathbf{u}_\mathcal{W} = \mathbf{0}$, we have $\mathbf{u} = \mathbf{0}$. Here, we'll need some linear algebra. The columns indexed by $\mathcal{W}$ are $|\mathcal{W}|$ linearly independent columns, so we know that $|\mathcal{W}| \leq m$ (because the columns are vectors in $\mathbb{R}^m$). Because A has full row rank, we know that we can extend $\mathcal{W}$ to some $\mathcal{B}$ (with $|\mathcal{B}| = m$) such that the columns indexed by $\mathcal{B}$ are still linearly independent, and therefore $\mathcal{B}$ is invertible.

Now, let $\mathcal{N}$ be the complement of $\mathcal{B}$. Because we found $\mathcal{B}$ by starting with $\mathcal{W}$ and possibly making it bigger, we know that $\mathcal{N}$ is found by starting with $\mathcal{Z}$ and possibly making it smaller. Because $\mathbf{x}_\mathcal{Z} = \mathbf{0}$ (that's how we chose $\mathcal{Z}$), we know that $\mathbf{x}_\mathcal{N} = \mathbf{0}$ as well.

Now we're nearly done. $A\mathbf{x} = A_\mathcal{B}\mathbf{x}_\mathcal{B} + A_\mathcal{N}\mathbf{x}_\mathcal{N} = \mathbf{b}$. Since $\mathbf{x}_\mathcal{N} = \mathbf{0}$, we have $A_\mathcal{B}\mathbf{x}_\mathcal{B} = \mathbf{b}$, so $\mathbf{x}_\mathcal{B} = A_\mathcal{B}^{-1}\mathbf{b}$. This is exactly what we wanted from a basic solution.

Second, suppose such a $\mathbf{u}$ exists. Then we have

$$A\mathbf{u} = A_\mathcal{W}\mathbf{u}_\mathcal{W} + A_\mathcal{Z}\mathbf{u}_\ddagger = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

This means that points of the form $\mathbf{x} + t\mathbf{u}$ still satisfy the system of equations: that is, $A(\mathbf{x} + t\mathbf{u}) = \mathbf{b} + t\mathbf{0} = \mathbf{b}$.

Not all points of the form $\mathbf{x} + t\mathbf{u}$ are feasible: for some t, we can find a coordinate i such that $x_i + tu_i < 0$. However, we know that such an i must be in $\mathcal{W}$, not $\mathcal{Z}$, because for any $i \in \mathcal{Z}$, we'd have $x_i = u_i = 0$. So $x_i > 0$. This means that when $|t|$ is sufficiently small, $x_i + tu_i > 0$ as well.

Choose a small enough $t > 0$ that $\mathbf{x} + t\mathbf{u}$ and $\mathbf{x} - t\mathbf{u}$ are both feasible: $x_i + tu_i > 0$ and $x_i - tu_i > 0$ for each i. This is a lot like pivoting: it's enough to ask that $|t| < \frac{|x_i|}{|u_i|}$ for each i such that $u_i \neq 0$.

But now, $\mathbf{x}$ lies on the line segment from $\mathbf{x} + t\mathbf{u}$ to $\mathbf{x} - t\mathbf{u}$. In fact, it's the midpoint of that line segment. Because $\mathbf{u} \neq \mathbf{0}$, and $t > 0$, it's distinct from both endpoints. So $\mathbf{x}$ is not an extreme point in this case, contrary to assumption. ☐