The Bipartite Matching Problem

Math 482, Lecture 21

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Bipartite graph

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We could write out X, Y, E as lists:

- $X = \{1, 2, 3\}$ and $Y = \{4, 5, 6, 7\}$.
- $E = \{(1, 4), (1, 6), (2, 5), (2, 7), (3, 4), (3, 5)\}$.

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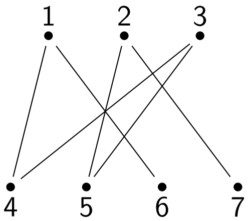
We could also draw a picture:



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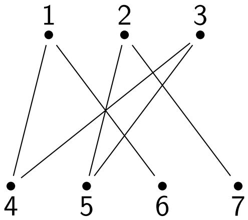

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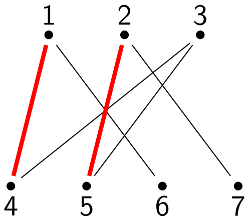

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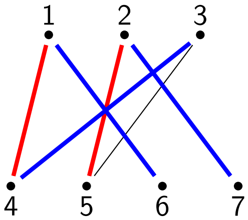

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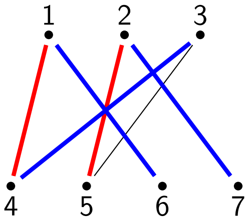

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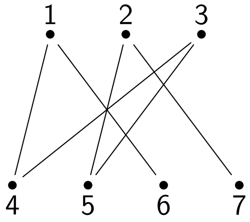
Bipartite Matching LP



$$
\begin{array}{ll}
\text{maximize} & x_{14} + x_{16} + x_{25} + x_{27} + x_{34} + x_{35} \\
\text{subject to} & x_{14} + x_{16} \leq 1 \\
& x_{25} + x_{27} \leq 1 \\
& x_{34} + x_{35} \leq 1 \\
& x_{14} + x_{34} \leq 1 \\
& x_{25} + x_{35} \leq 1 \\
& x_{16} \leq 1 \\
& x_{27} \leq 1 \\
& x_{14}, x_{16}, x_{25}, x_{27}, x_{34}, x_{35} \geq 0
\end{array}
$$

Idea: $x_{ij} = 1$ if (i, j) is in the matching, and $x_{ij} = 0$ otherwise.

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- But it looks like there might be a problem. What if the optimal solution sets, for example, $x_{14} = \frac{1}{2}$? We can't interpret this as a matching!
- Enforcing the constraint that x_{ij} is an integer ($x_{ij} = 0$ or $x_{ij} = 1$) is hard. (We'll talk about this later in the class.)
- The bipartite matching LP has a special property that guarantees integer optimal solutions, without having to explicitly ask for it.

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If the $m \times n$ matrix A is TU and $\mathbf{b} \in \mathbb{R}^m$ is an integer vector, then all corner points of $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ have integer coordinates.

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On Friday, we will see that for the bipartite matching LP, the constraint matrix A is always TU. This explains why we don't have to worry about integrality!

Big idea #1: integer inverses

How do we find a basic solution of $A\mathbf{x} = \mathbf{b}$?

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If we can guarantee that the inverse matrix $A_{\mathcal{B}}^{-1}$ has integer entries, then $\mathbf{x}_{\mathcal{B}}$ will always be an integer, too.

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$\implies$: We need $\det(M) = \pm 1$, otherwise $\det(M^{-1}) = \frac{1}{\det(M)}$ won't be an integer (and M^{-1} can't have all integer entries).

$\impliedby$: There is a formula for M^{-1} in which the denominator is $\det(M)$. E.g., for 3×3 matrices,

$$M^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{\det(M)} \begin{bmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{bmatrix}$$

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  Because we have a system of inequalities $A\mathbf{x} \leq \mathbf{b}$, which gives us the bigger system of equations $A\mathbf{x} + I\mathbf{s} = \mathbf{b}$. If we take k columns from A and $m - k$ columns from I to build $A_{\mathcal{B}}$, the determinant will equal the determinant of a smaller $k \times k$ submatrix of A.

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- Why does the TU condition allow determinants to be 0 in addition to ± 1?

  Not all choices of $\mathcal{B}$ are a valid basis: sometimes $\det(A_{\mathcal{B}}) = 0$, and $A_{\mathcal{B}}^{-1}$ does not exist. But if this happens, that's fine. If some basis $\mathcal{B}$ doesn't give any basic solution, in particular it does not give a fractional basic solution.