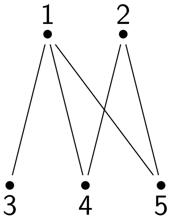
The Bipartite Matching Problem II

Math 482, Lecture 22

Misha Lavrov

March 27, 2020

Last time: bipartite matching LP



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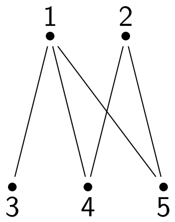

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- Variables: x_{ij} for every edge $(i, j) \in E$.

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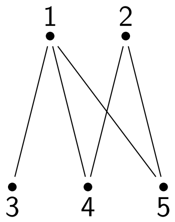

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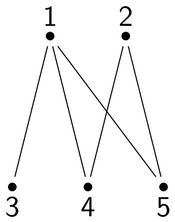

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- Variables: x_{ij} for every edge $(i, j) \in E$.
- Maximize sum of all variables.
- For every vertex $i \in X \cup Y$, sum of variables involving i is ≤ 1.

Incidence matrix

$$\begin{aligned} \text{maximize} \quad & x_{13} + x_{14} + x_{15} + x_{24} + x_{25} \\ \text{subject to} \quad & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{14} \\ x_{15} \\ x_{24} \\ x_{25} \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ & x_{13}, x_{14}, x_{15}, x_{24}, x_{25} \geq 0 \end{aligned}$$

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- In general, constraints are $A\mathbf{x} \le \mathbf{1}$.
- A has $|X| + |Y|$ rows and $|E|$ columns.
- A is the *incidence matrix* of the bipartite graph: $A_{v,e} = 1$ if vertex v is an endpoint of edge e, and 0 otherwise.

Totally unimodular matrices

Previous lecture:

Definition

A matrix A is **totally unimodular** (TU for short) if every square submatrix (any k rows and any k columns, not necessarily consecutive, for all values of k) has determinant -1, 0, or 1.

Theorem

If the $m \times n$ matrix A is TU and $\mathbf{b} \in \mathbb{R}^m$ is an integer vector, then all corner points of $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ have integer coordinates.

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   - All columns of B have two 1s: $\det(B) = 0$.
3. By induction on k, all submatrices have determinant 0 or ± 1.

Case 1: B has a column of all zeroes

Example:

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & 1 & \mathbf{0} & 0 \\ 0 & 0 & 0 & 1 & 1 \\ \mathbf{1} & \mathbf{0} & 0 & \mathbf{0} & 0 \\ 0 & 1 & 0 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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If B has a column of all zeroes, then the columns of B are linearly dependent. In that case, $\det(B) = 0$.

Case 2: B has a column with only one 1

Example:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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If B has a column with only one 1, expand $\det(B)$ along that column.

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If B has a column with only one 1, expand $\det(B)$ along that column. Reduce to a smaller matrix:

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

Case 3: All columns of B have two 1s

Example:

$$\begin{bmatrix} 1 & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

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In the final case, rows of B are linearly dependent:

- The rows coming from X add up to $\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$.
- The rows coming from Y *also* add up to $\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$.

Therefore $\det(B) = 0$.

Taking the dual

What is the dual of the bipartite matching LP?

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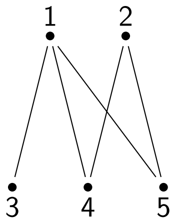
- The primal problem has a variable $x_{ij} \geq 0$ for every edge. So, the dual has a $\geq$ constraint for every edge $(i, j) \in E$.
- The primal LP is a maximization problem. So, the dual LP is a minimization problem: we minimize the sum of all the y_i.
- The primal variable x_{ij} appears in constraints for vertices i and j. So, the dual constraint for (i, j) contains variables y_i and y_j: we get
$$y_i + y_j \geq 1 \quad \text{for each } (i, j) \in E$$

An example of the dual LP



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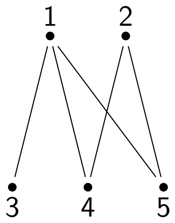

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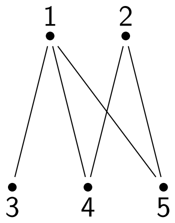

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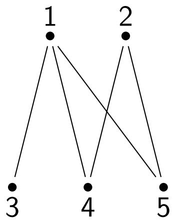

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Let's interpret the dual LP! Let S be the set of all i such that $y_i = 1$.

- Want to minimize the size of S.
- For each $(i, j) \in E$, either $i \in S$ or $j \in S$ (or both).

Vertex covers

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A **vertex cover** in a graph is a set of vertices S that includes at least one endpoint of every edge.

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Theorem

In any bipartite graph, **the number of edges in a maximum matching** *is equal to* **the number of vertices in a minimum vertex cover***.*

Proof.

Linear programming duality. □

General graphs

We can look at both of these problems in graphs that are not bipartite. Such a graph also has vertices and edges, but the vertices don't have two types X and Y.

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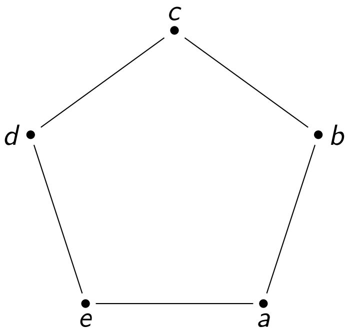

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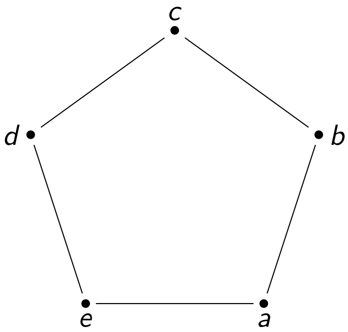

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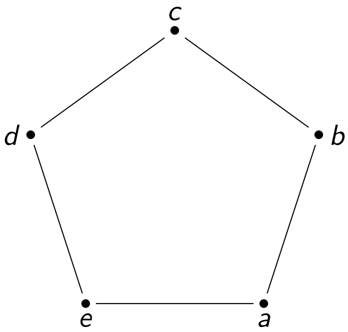

- Vertices $\{a, b, c, d, e\}$ and edges $\{ab, bc, cd, de, ae\}$.
- One largest matching: $\{ab, cd\}$.
- One smallest vertex cover: $\{a, c, d\}$.

Strong duality fails!

For this non-bipartite graph, the theorem doesn't work: the largest matching is smaller than the smallest vertex cover! What went wrong?

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- Can still write down LPs for the largest matching and the smallest vertex cover.
- These LPs are still dual and still have the same objective value.
- The constraint matrix is not totally unimodular! So the optimal solutions of the two LPs might be fractional, and not actually give a matching or a vertex cover!

  In the example on the last slide: both LPs have an objective value of 2.5.