The Max-Flow Min-Cut Theorem

Math 482, Lecture 24

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The max-flow min-cut theorem

Last time, we proved that for any network:

Theorem

If $\mathbf{x}$ *is a feasible flow, and* (S, T) *is a cut, then*

$$v(\mathbf{x}) \leq c(S, T):$$

the value of $\mathbf{x}$ *is at most the capacity of* (S, T).

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The plan for today is to prove, additionally:

Theorem

If $\mathbf{x}$ is a maximum flow and (S, T) is a minimum cut ($\mathbf{x}$ maximizes $v(\mathbf{x})$ and (S, T) minimizes $c(S, T)$), then

$$v(\mathbf{x}) = c(S, T).$$

Proof ingredients

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- **Step 1:** We will show that the duality between flows and cuts is exactly LP duality. That is, we’ll take the dual of the max-flow problem, and show that it is the min-cut problem.

Proof ingredients

- What we proved last time was the "weak duality" version. What we're proving is analogous to "strong duality".
- **Step 1:** We will show that the duality between flows and cuts is exactly LP duality. That is, we'll take the dual of the max-flow problem, and show that it is the min-cut problem.
- **Step 2:** For flows, there are no problems with integrality. But cuts will correspond to *integer* solutions of the dual LP.

  We will prove that this is not a problem by showing that the constraint matrix is TU.

The maximum flow LP

Last time, we wrote down the LP for maximum flow:

$$\begin{aligned}
\underset{\mathbf{x}}{\text{maximize}} \quad & \sum_{j:(s,j)\in A} x_{sj} \\
\text{subject to} \quad & \sum_{i:(i,k)\in A} x_{ik} - \sum_{j:(k,j)\in A} x_{kj} = 0 \quad (k \in N, k \neq s, t) \\
& x_{ij} \leq c_{ij} \quad (i,j) \in A \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}$$

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- A dual variable u_k for each node $k \in N$ other than s, t.

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(We assume there are no arcs into s or out of t.)

We see that the dual will have:

- A dual variable u_k for each node $k \in N$ other than s, t.
- A dual variable y_{ij} for each arc $(i,j) \in A$.

The dual LP

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$$\begin{aligned} \underset{\mathbf{u},\mathbf{y}}{\text{minimize}} \quad & \sum_{(i,j)\in A} c_{ij} y_{ij} \\ \text{subject to} \quad & -u_i + u_j + y_{ij} \ge 0 \qquad (x_{ij}, i \ne s, j \ne t) \end{aligned}$$

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& \mathbf{y} \ge \mathbf{0}, \mathbf{u} \text{ unrestricted} &
\end{aligned}$$

Simplifying the dual LP

The dual constraints all have nearly the same form, except for constraints corresponding to nodes s and t:

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\begin{aligned}
-u_i + u_j + y_{ij} &\geq 0 && \text{most arcs } (i,j) \\
u_j + y_{sj} &\geq 1 && \text{arcs } (s,j) \\
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The dual LP as a minimax problem

Our current dual LP:

$$\begin{aligned} \underset{\mathbf{u},\mathbf{y}}{\text{minimize}} \quad & \sum_{(i,j)\in A} c_{ij} y_{ij} \\ \text{subject to} \quad & y_{ij} \geq u_i - u_j && (i,j) \in A \\ & u_s = 1, u_t = 0 \\ & \mathbf{y} \geq \mathbf{0}, \mathbf{u} \text{ unrestricted} \end{aligned}$$

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The only constraints on y_{ij} are lower bounds: $y_{ij} \geq u_i - u_j$ and $y_{ij} \geq 0$.

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  So the objective function is exactly $\sum_{i\in S} \sum_{j\in T} c_{ij} = c(S, T)$.

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Will all optimal solutions have this form?

Finishing the proof

Our goal is to show that all optimal solutions to the dual LP correspond to cuts. This will complete the proof of min-cut max-flow by strong LP duality.

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- In any optimal solution, $u_k \in [0, 1]$ for all nodes k.

  (This is what we will do next.)

Showing that $u_k \in [0, 1]$

Lemma

The linear program

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has an optimal solution $\mathbf{u}$ *in which* $0 \le u_k \le 1$ *for all* k.

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has an optimal solution $\mathbf{u}$ *in which* $0 \leq u_k \leq 1$ *for all* k.

To prove this, take an optimal solution $\mathbf{u}$.

Replace each u_k by $\max\{0, \min\{1, u_k\}\}$, "clipping" u_k to $[0, 1]$.

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Replace each u_k by $\max\{0, \min\{1, u_k\}\}$, "clipping" u_k to $[0, 1]$.

Want to show: when we do this, $\max\{0, u_i - u_j\}$ never increases.

Casework

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- Suppose $u_i > 1$:

  Then $u_i - u_j$ decreases by $(1 - u_i)$.

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- Suppose $u_j < 0$:

  Then $u_i - u_j$ decreases by $|u_j|$.

Finally, if $0 \leq u_j < u_i \leq 1$, nothing changes.

Total unimodularity

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The constraint matrix of

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2. Check matrices with a column with ≤ 1 nonzero entry.

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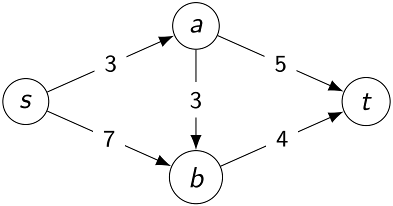
1. Check 1×1 matrices.
2. Check matrices with a column with ≤ 1 nonzero entry.
3. Deal with exceptional cases.

Total unimodularity example



Constraints:

$$
\begin{aligned}
y_{sa} \quad & - u_s + u_a & \geq 0 \\
y_{sb} \quad & - u_s + u_b & \geq 0 \\
y_{ab} \quad & - u_a + u_b & \geq 0 \\
y_{at} \quad & - u_a + u_t & \geq 0 \\
y_{bt} \quad & - u_b + u_t & \geq 0
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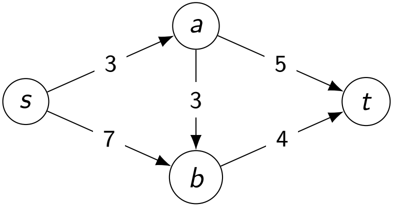

Constraints:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Exceptional case

Only kind of submatrix that has no row or column with ≤ 1 nonzero entry:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \mathbf{-1} & \mathbf{1} & \mathbf{0} & 0 \\ 0 & 1 & 0 & 0 & 0 & \mathbf{-1} & \mathbf{0} & \mathbf{1} & 0 \\ 0 & 0 & 1 & 0 & 0 & \mathbf{0} & \mathbf{-1} & \mathbf{1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

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In general: this happens if, whenever we pick the row for arc (i, j), we pick both the u_i and u_j columns.

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$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

In general: this happens if, whenever we pick the row for arc (i, j), we pick both the u_i and u_j columns.

In this case, the determinant is 0: the columns add to 0, so they are linearly dependent.

Exceptional case

Only kind of submatrix that has no row or column with ≤ 1 nonzero entry:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

In general: this happens if, whenever we pick the row for arc (i, j), we pick both the u_i and u_j columns.

In this case, the determinant is 0: the columns add to 0, so they are linearly dependent.

This completes the proof of the min-cut max-flow theorem.