Augmenting Paths

Math 482, Lecture 25

Misha Lavrov

April 3, 2020

Lecture plan

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1. Describe a simple greedy algorithm that tries to find a max flow.
2. See it get stuck.
3. Make the algorithm more powerful.

Directed s, t-paths

Definition

In a network, a *directed path* from s to t is a sequence

$$s, v_1, v_2, \ldots, v_k, t$$

where $v_1, v_2, \ldots, v_k \in N$ and $(s, v_1), (v_1, v_2), \ldots, (v_k, t) \in A$.

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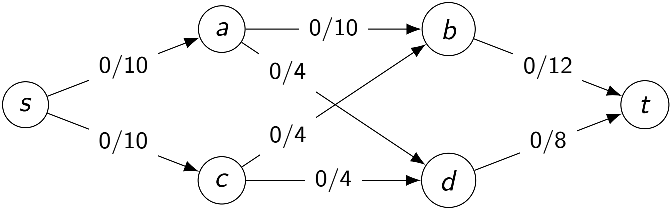
Example:



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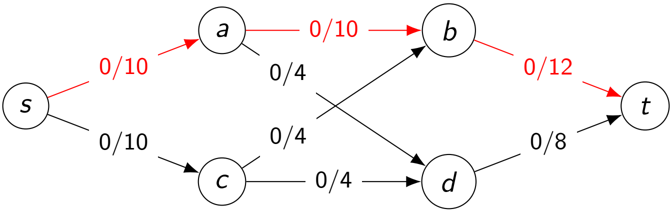
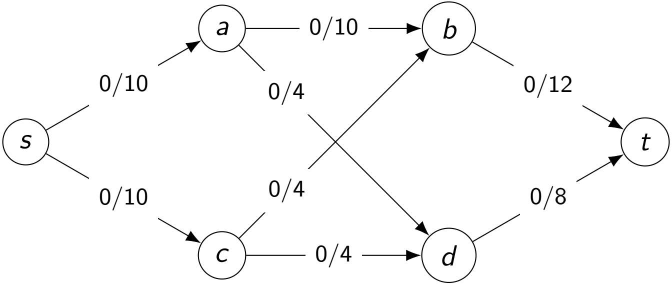
Example: directed path $s \to a \to b \to t$



Using a directed path

Whenever we have a directed path from s to t and all arcs along the path are below capacity, we can use it to increase the flow.



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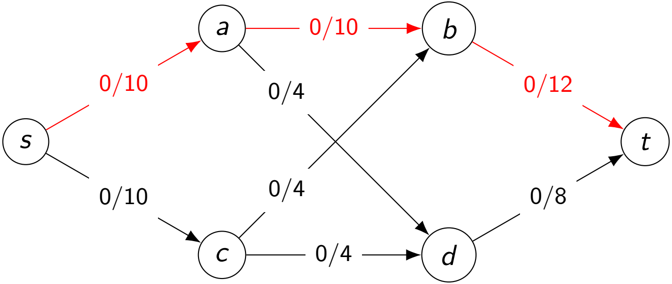

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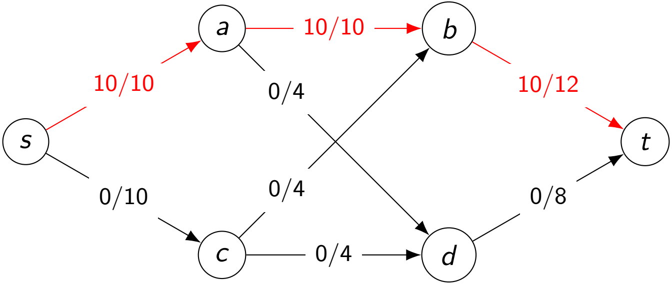

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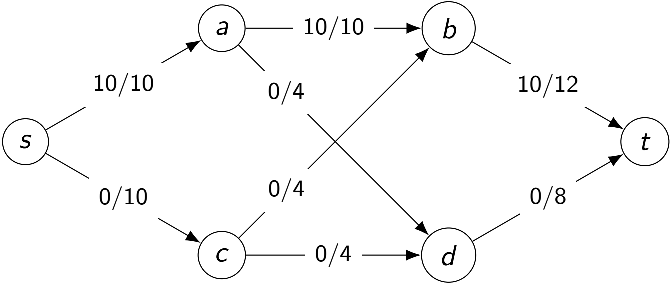

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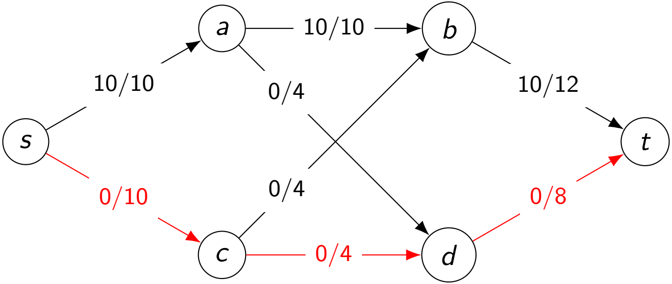

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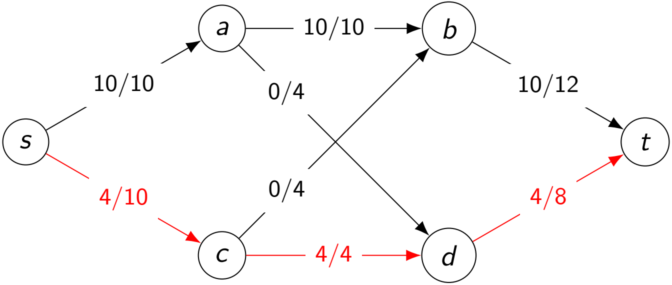

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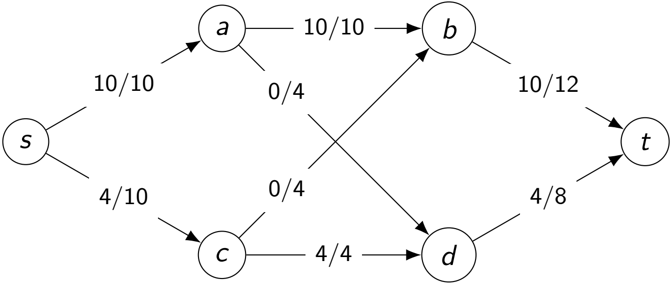

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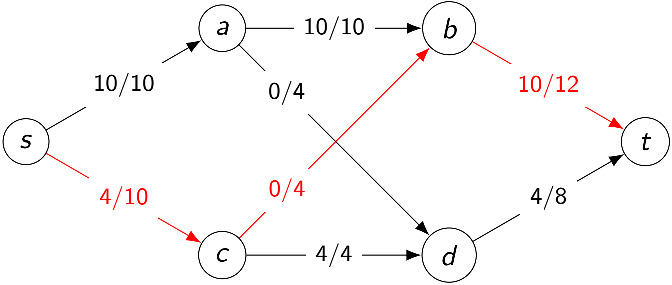

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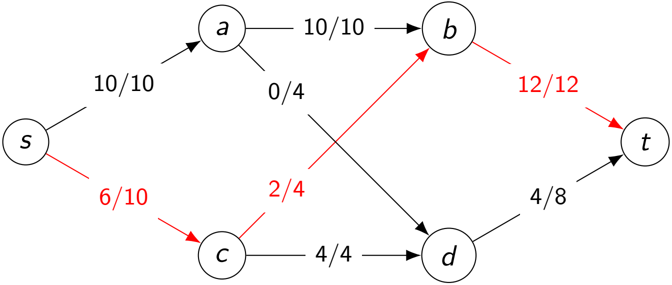

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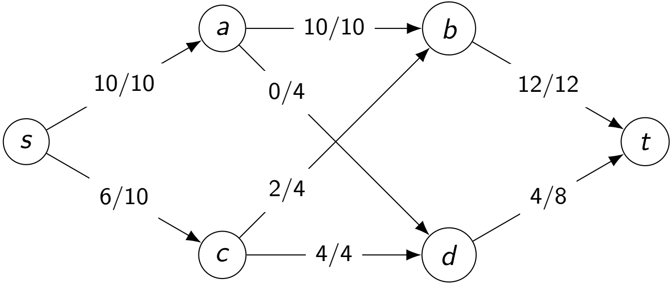
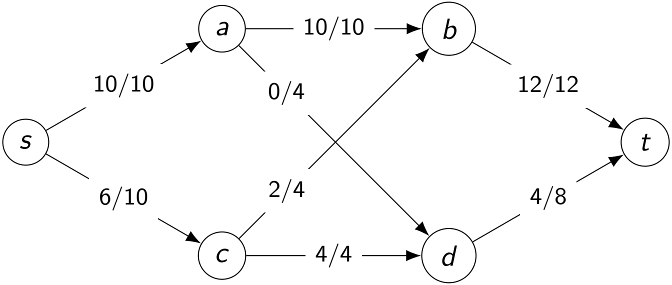


At this point, there are no more directed paths where all arcs are below capacity. But is this the maximum flow?

A further improvement

If we redirect some $a \to b \to t$ flow to go $a \to d \to t$, we can send more flow along the path $s \to c \to d \to t$...



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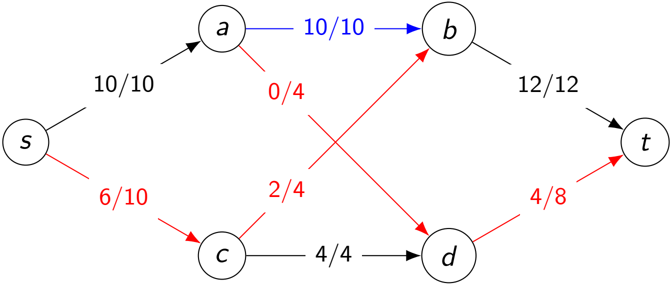

Increase flow along red edges, decrease flow along blue edge.

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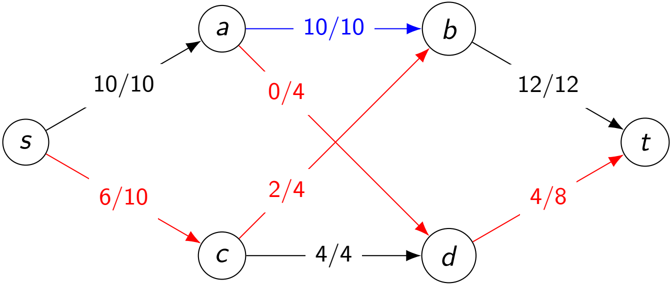

Increase flow along red edges, decrease flow along blue edge.

Note: $s \to c \to b \leftarrow a \to d \to t$ is *almost* a directed path.

Augmenting paths

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Given a network (N, A) and a feasible flow $\mathbf{x}$, an augmenting path for $\mathbf{x}$ is a sequence of nodes

$$s = v_0, v_1, v_2, \ldots, v_k, v_{k+1} = t$$

such that for each pair v_i, v_{i+1}:

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The sequence

$$s \to c \to b \leftarrow a \to d \to t$$

we found on the previous slide was an augmenting path.

Using an augmenting path to improve **x**

To augment a feasible flow **x** along an augmenting path:

1. Let $\delta > 0$ be the largest value such that
   - $x_e \leq c_e - \delta$ for all forward arcs on the path;
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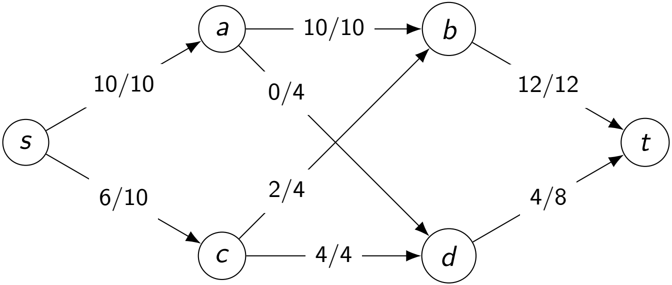
When we do this, flow is still conserved at internal nodes of the augmenting path. There are four possible cases:

$$\cdots \xrightarrow{+\delta} p \xrightarrow{+\delta} \cdots \qquad \cdots \xrightarrow{+\delta} q \xleftarrow{-\delta} \cdots$$

$$\cdots \xleftarrow{-\delta} r \xrightarrow{+\delta} \cdots \qquad \cdots \xleftarrow{-\delta} s \xleftarrow{-\delta} \cdots$$

Example of augmenting



1. Find the augmenting path.

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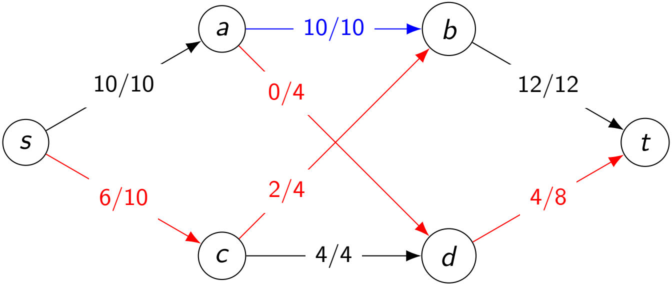

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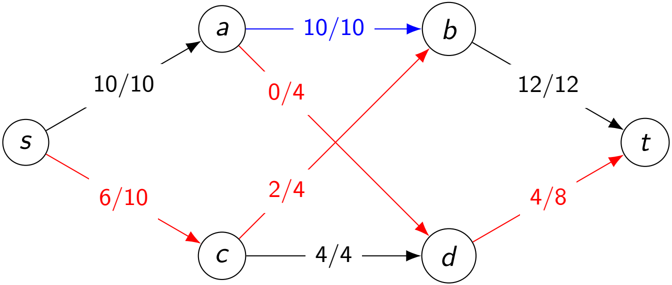

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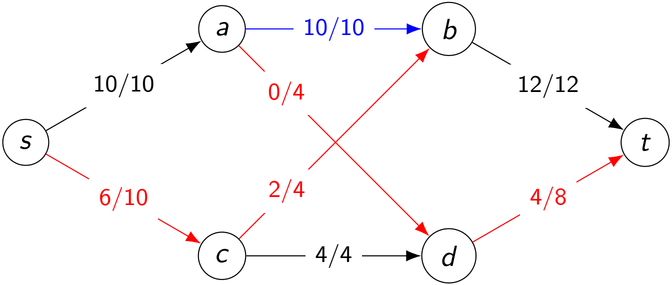

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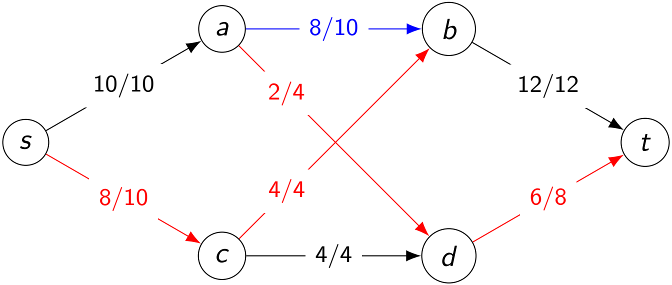

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3. Increase or decrease the flow along each edge by δ.

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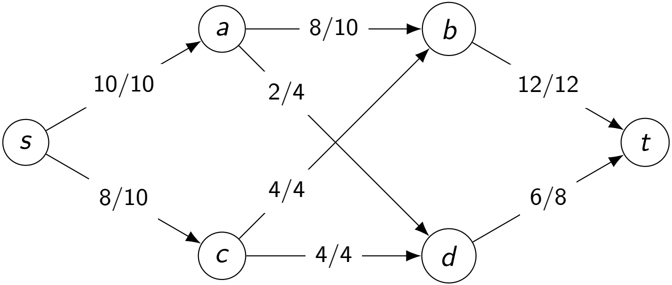

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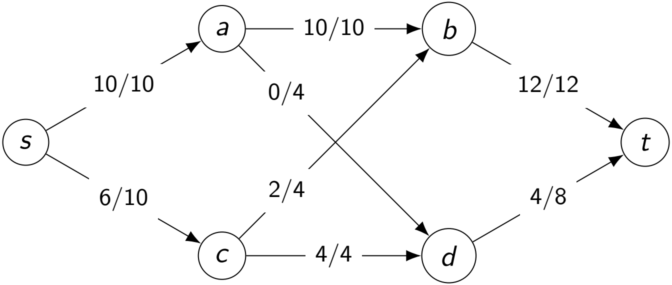
Idea: augmenting paths for $\mathbf{x}$ are directed paths in the residual graph.

Residual graph example

We construct the residual graph for our next-to-last feasible flow:



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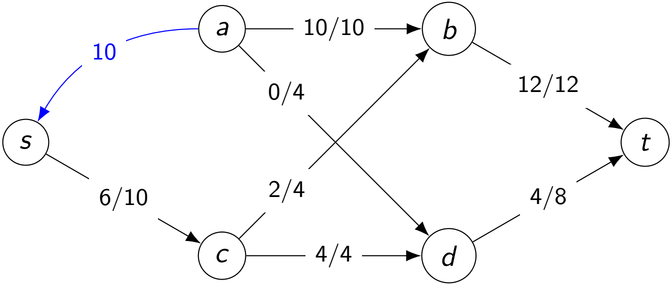

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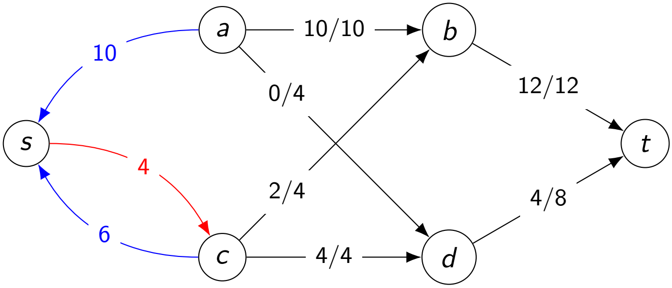

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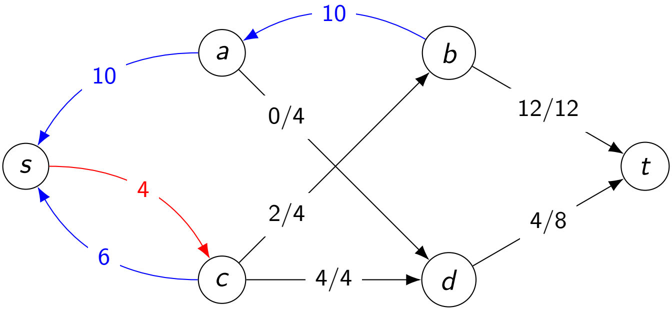

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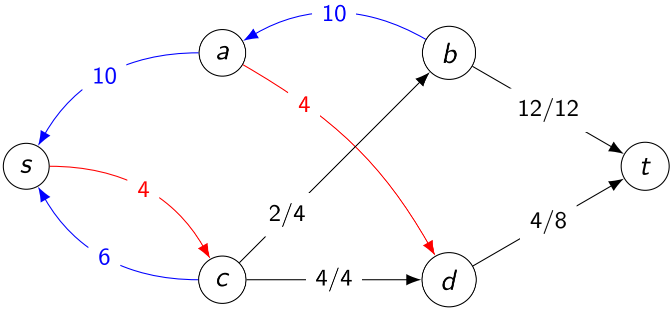

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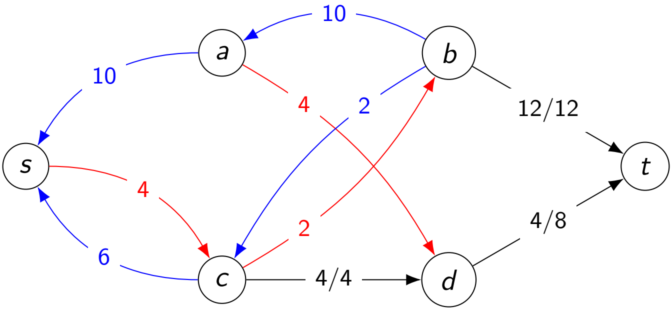

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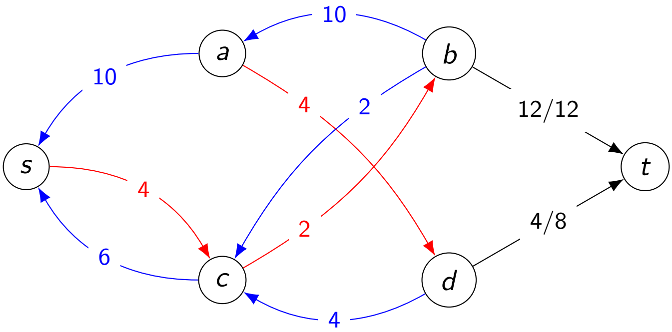

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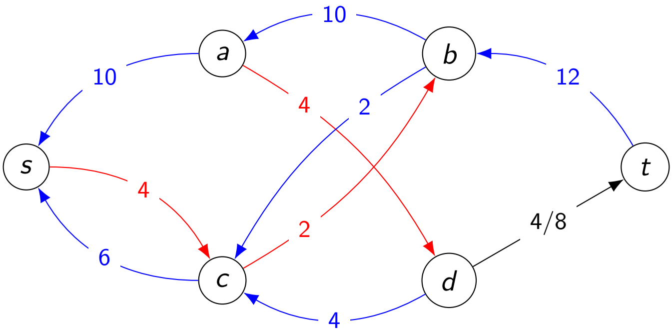

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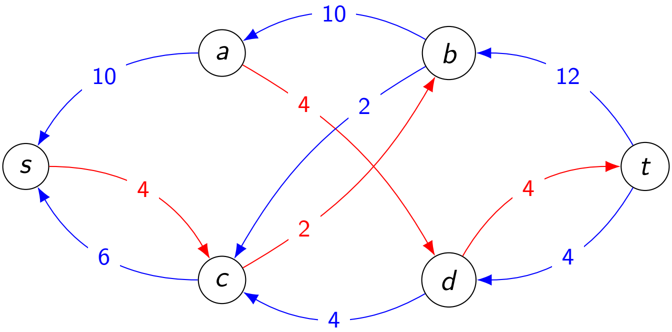

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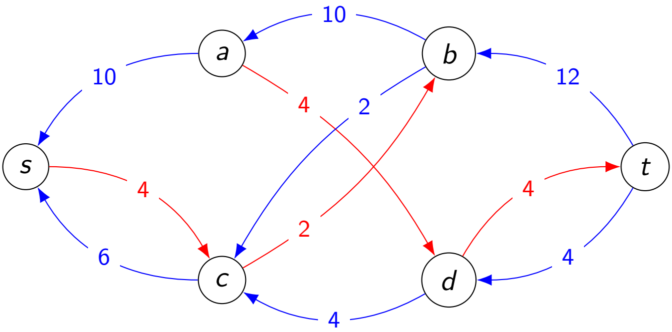

We can find the augmenting path $s \to b \leftarrow a \to d \to t$ by following only *forward* arrows (of either color) in the residual graph; $\delta = 2$ is the smallest number along those arrows.