Variations on Max-Flow Problems

Math 482, Lecture 27

Misha Lavrov

April 8, 2020

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Problems we'll consider:

- Supply and demand problems.
- Finding feasible flows with lower and upper bounds.
- Bipartite matchings and vertex covers (again).

Posing a supply/demand problem

Recall: given a network (N, A), a flow $\mathbf{x}$, and a node $k \in N$, the *excess at k* is

$$\Delta_k(\mathbf{x}) := \sum_{i:(i,k)\in A} x_{ik} - \sum_{j:(k,j)\in A} x_{kj}.$$

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- When $d_k < 0$, k is a *supply node*: it has extra flow it wants to get rid of.
- When $d_k > 0$, k is a *demand node*: it wants more entering than leaving flow.

Properties of supply/demand problems

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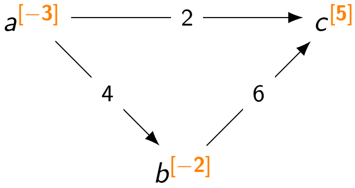

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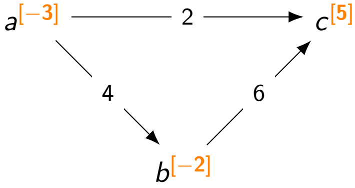

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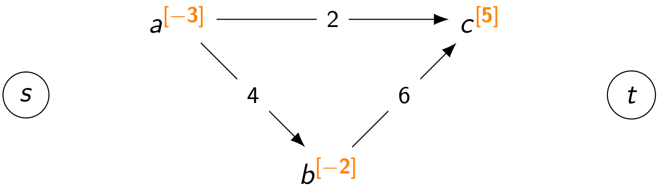

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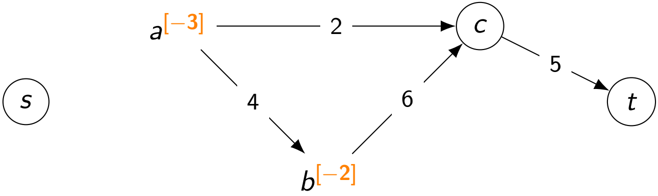

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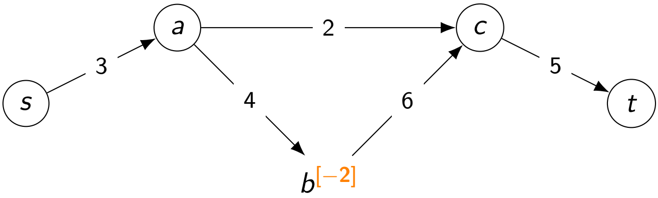

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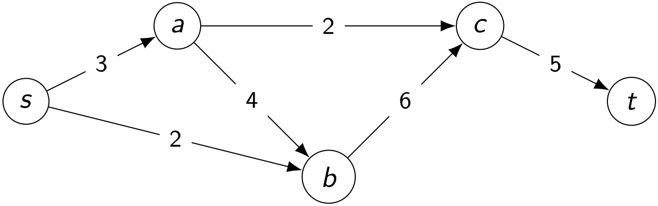

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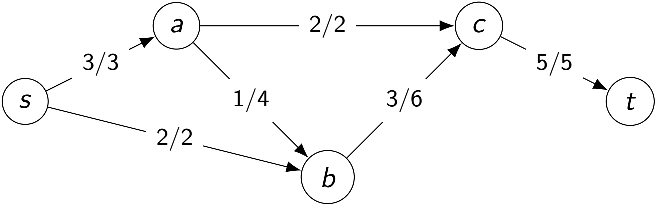

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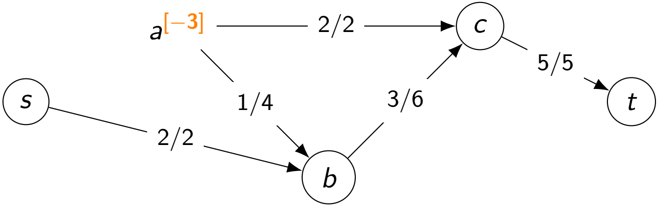

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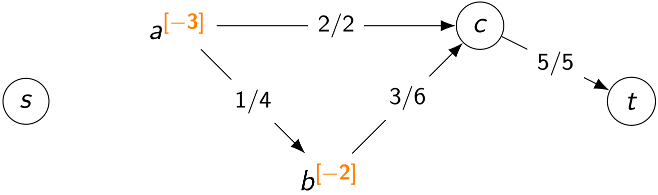

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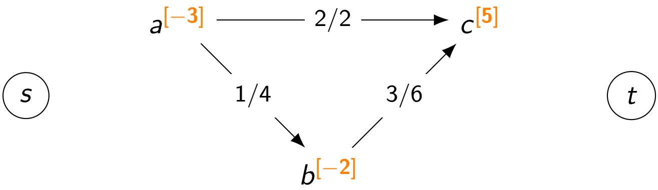

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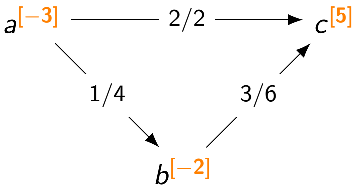

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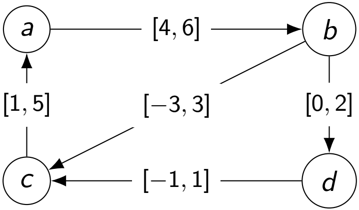

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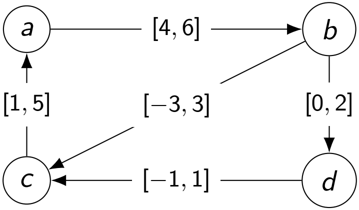
4. In the end, each node k has

$$d_k = \sum_{j:(k,j)\in A} a_{kj} - \sum_{i:(i,k)\in A} a_{ik}.$$

(We could skip directly to this step.)

Example of the reduction

Here is an example:



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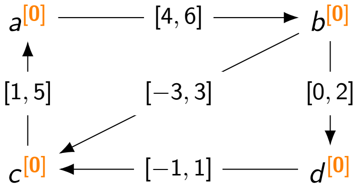

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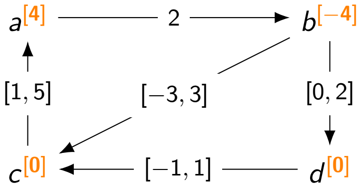

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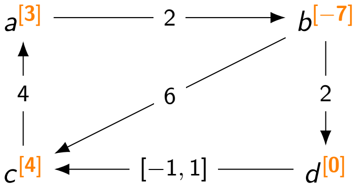

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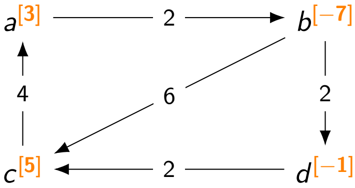

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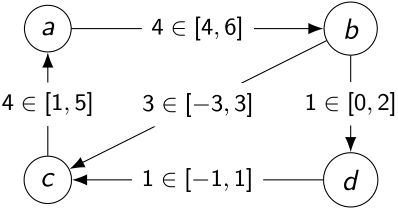

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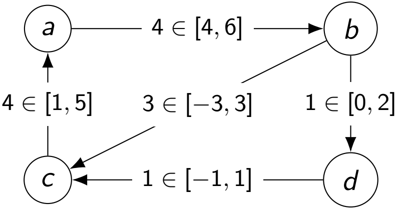

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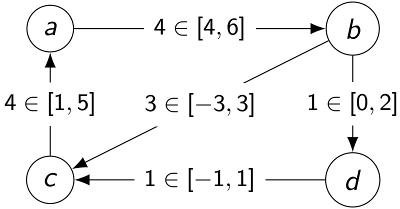

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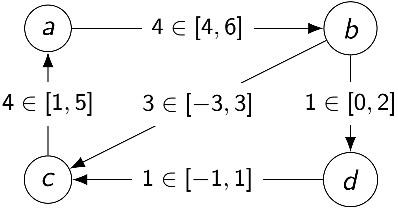

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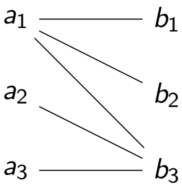
An maximum flow will send 1 flow along every edge in a matching, and 0 flow along all other edges.

Bipartite matching example

Going from the bipartite graph to a network:



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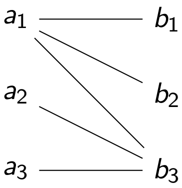

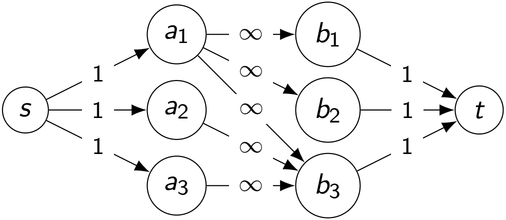

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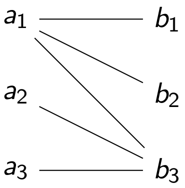
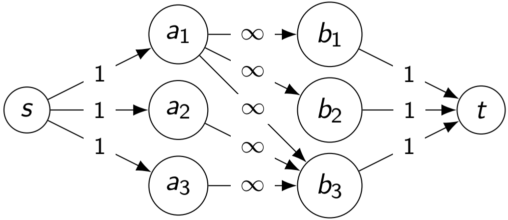
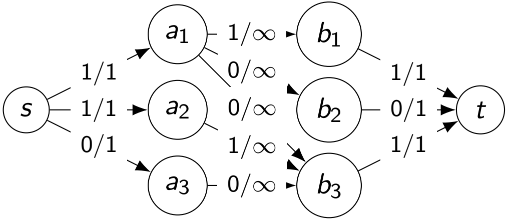


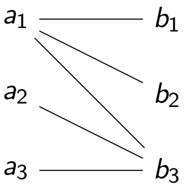


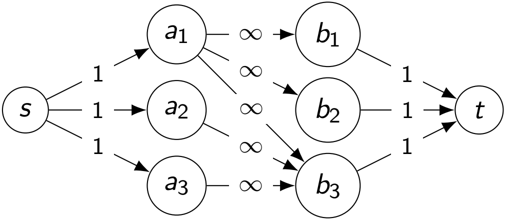
Going from a maximum flow to a maximum matching:



Bipartite matching example

Going from the bipartite graph to a network:





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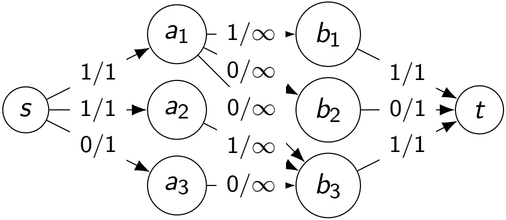
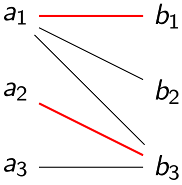
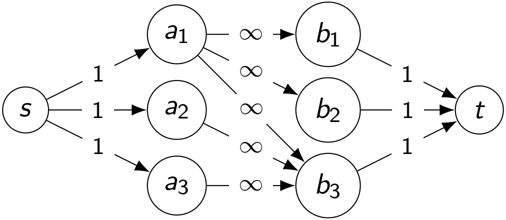




Minimum cuts and vertex covers

What does a cut look like in this picture?



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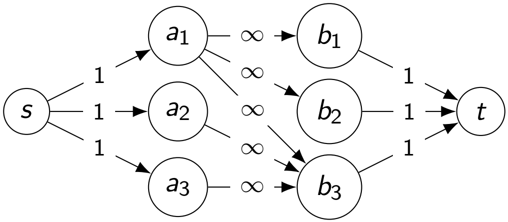

A cut is *really bad* if one of the ∞ arcs crosses the cut.

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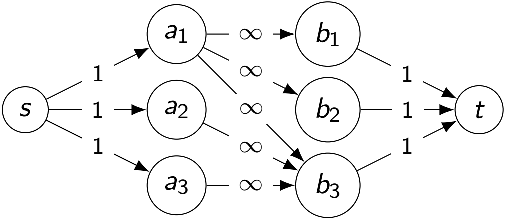

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So there are no arcs from $A \cap S$ to $B \cap T$.

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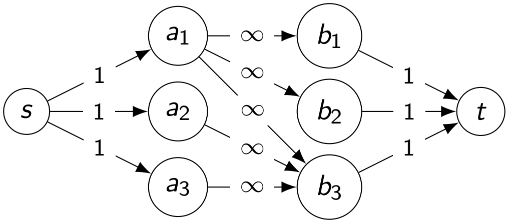

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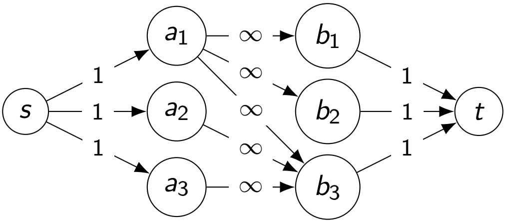

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So there are no arcs from $A \cap S$ to $B \cap T$.

So together, $A \cap T$ and $B \cap S$ form a vertex cover. We can check that the capacity of such a cut (S, T) is $|A \cap T| + |B \cap S|$: the number of vertices in the cover.