Minimum-Cost Flow

Math 482, Lecture 28

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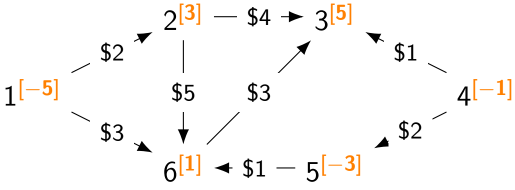
Example:



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- What a basic solution looks like.
- How to do a pivoting step.
- How to determine the reduced costs of an arc.

Number of basic variables

Our constraints are: $F\mathbf{x} = \mathbf{d}$, where

- F is a $|N| \times |A|$ matrix where x_{ij}'s column has a 1 in row j and a -1 in row i.
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So our basis will have $|N| - 1$ variables, assuming the network is connected.[1]

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[1] If there are two or more subnetworks with no arcs between them, we solve the subproblems separately.

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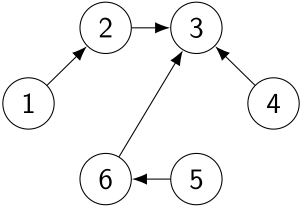

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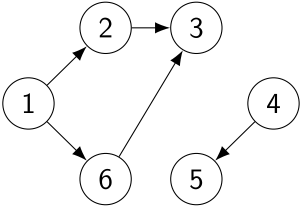

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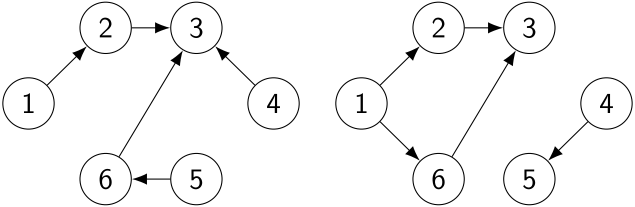

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Claim: $|N| - 1$ variables form a basis exactly when their arcs make a spanning tree.

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A2. We have an algorithm (next slide) to find a basic solution using only arcs in a spanning tree.

This will give us an $\mathbf{x}$ such that $F\mathbf{x} = \mathbf{d}$, but not necessarily $\mathbf{x} \geq \mathbf{0}$.

From a spanning tree to a basic solution

To find a basic solution, repeat the following:

1. Pick a node k with only one arc of the spanning tree with unknown flow in/out of k.
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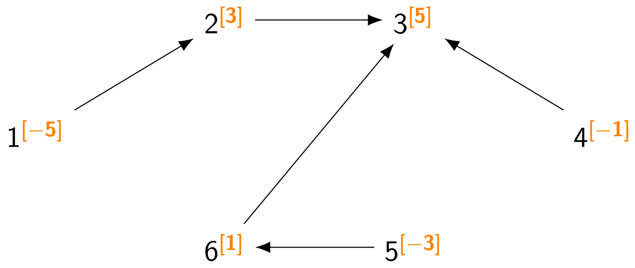

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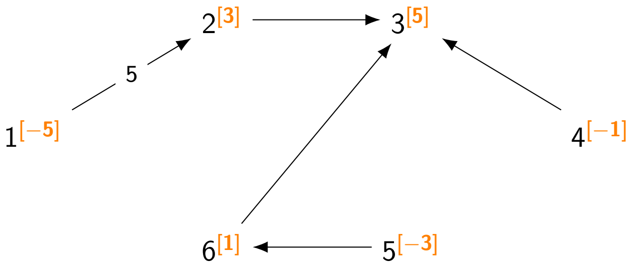

Solve for x_{12} using node 1

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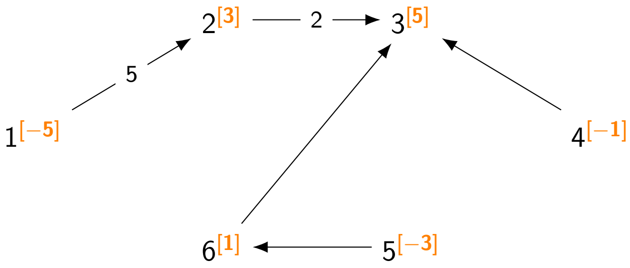

Solve for x_{23} using node 2

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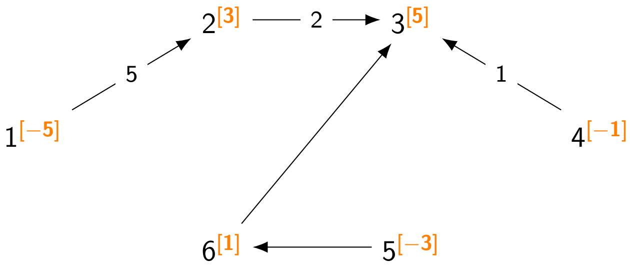

Solve for x_{43} using node 4

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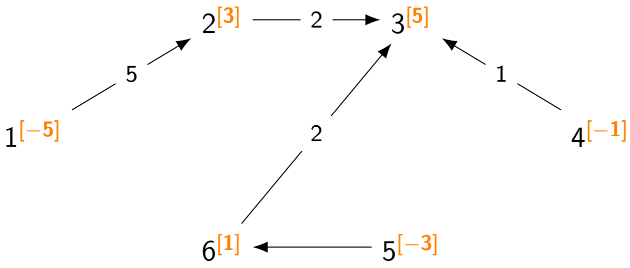

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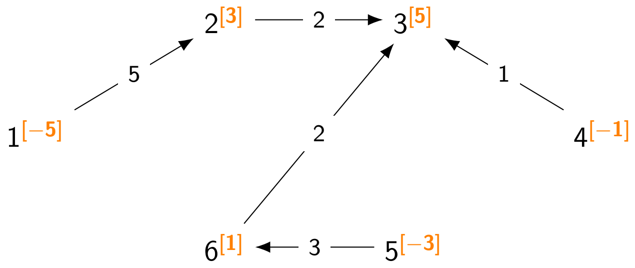

Solve for x_{56} using node 5

Ideas about pivoting steps

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One of those solutions will have flow $x_{ij} = 0$ for an existing arc (i, j).

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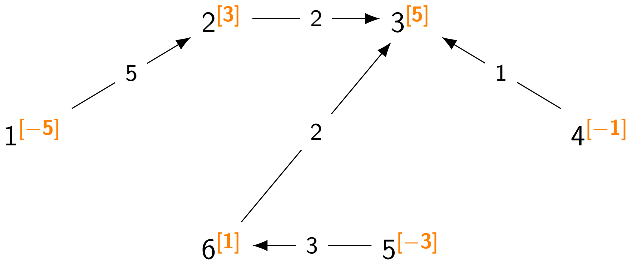

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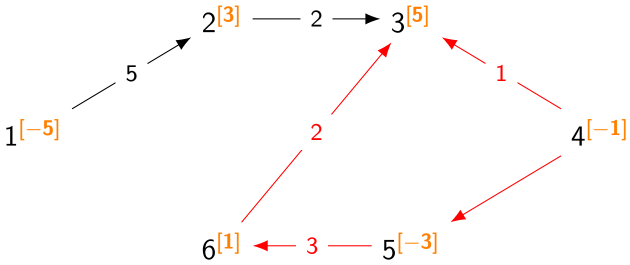

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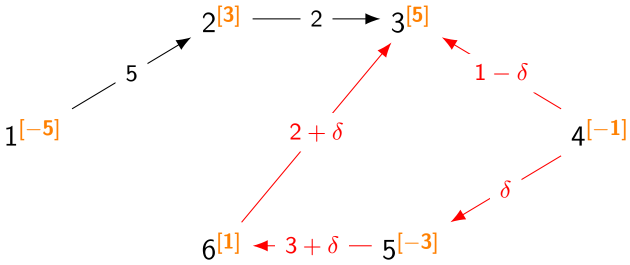

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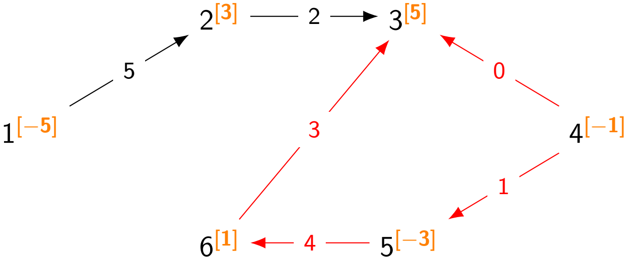

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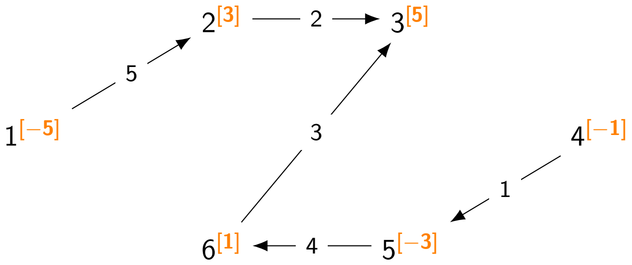

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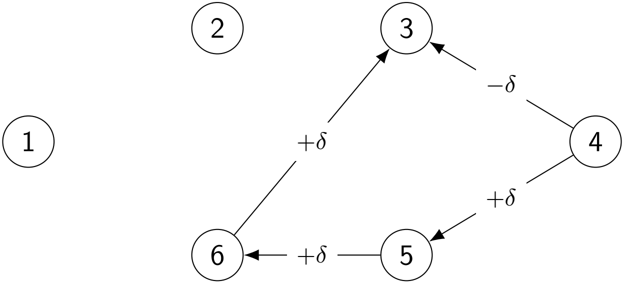

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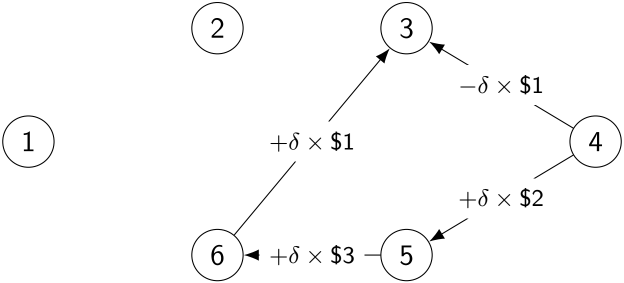

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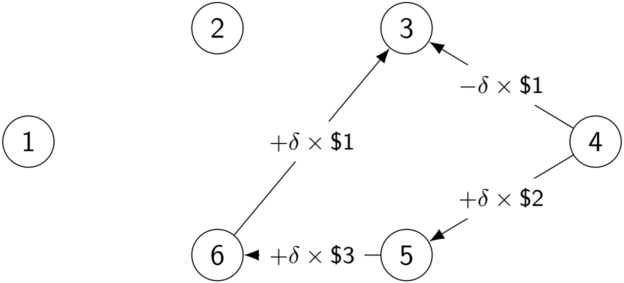

Total change in cost: $\delta \times (2 + 3 + 1 - 1) = 5\delta$, so the reduced cost of x_{45} is 5.

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   Set the cost to \$1 for artificial arcs, \$0 for original arcs.

Example of the first phase

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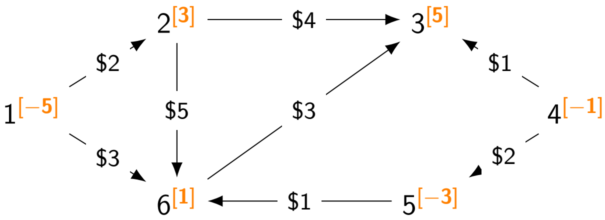

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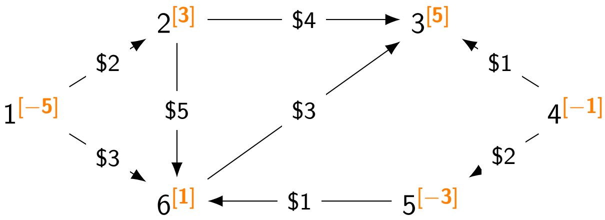
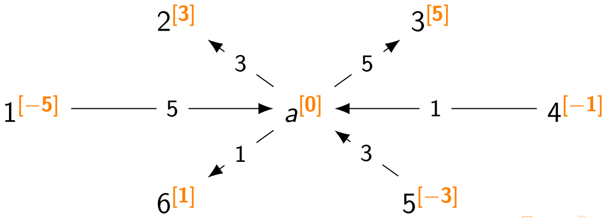


Initial spanning tree for the phase-one problem:



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Once we delete the artificial node and artificial arcs, we're left with a basic feasible solution to the original problem.