Integer programming

Math 482, Lecture 32

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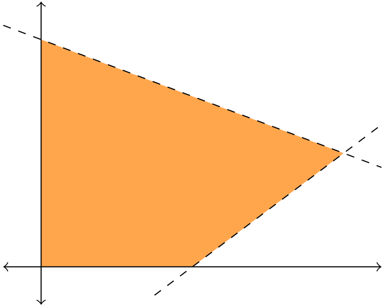
- Total unimodularity is important in integer programming, but doesn't often happen: usually, the integrality matters.

Some examples

Here is a completely ordinary linear program:

$$\begin{aligned} \underset{x,y\in\mathbb{R}}{\text{maximize}} \quad & x + y \\ \text{subject to} \quad & 3x + 8y \leq 24 \\ & 3x - 4y \leq 6 \\ & x, y \geq 0 \end{aligned}$$

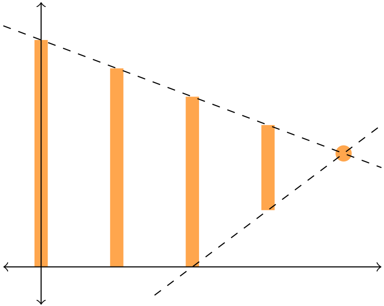


The optimal solution is $(x, y) = (4, \frac{3}{2})$.

Some examples

Now, change x to an integer variable:

$$\begin{aligned} \underset{x \in \mathbb{Z}, y \in \mathbb{R}}{\text{maximize}} \quad & x + y \\ \text{subject to} \quad & 3x + 8y \leq 24 \\ & 3x - 4y \leq 6 \\ & x, y \geq 0 \end{aligned}$$



The optimal solution is still $(x, y) = (4, \frac{3}{2})$. Coincidentally, the integrality didn't matter.

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Now, make x and y both integers:

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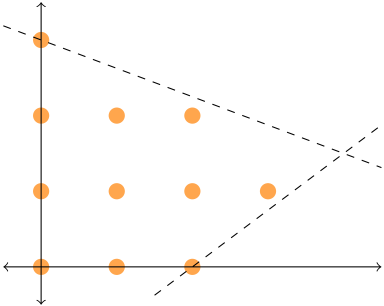

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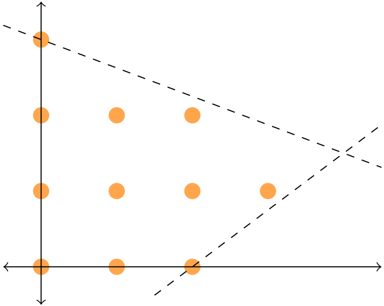

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Note that rounding $(4, \frac{3}{2})$ to the nearest integer won't give us an optimal or even feasible solution!

Difficulty of approximation

Optimal integer solutions can be arbitrarily far from optimal real solutions. Example: take the region

$$\left\{(x, y) \in \mathbb{R} : \frac{x-1}{998} \leq y \leq \frac{x}{1000}, x \geq 0, y \geq 0\right\}.$$

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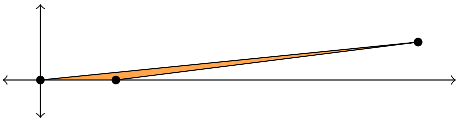

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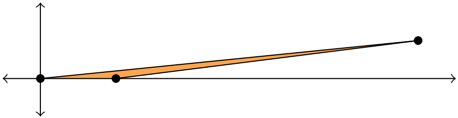

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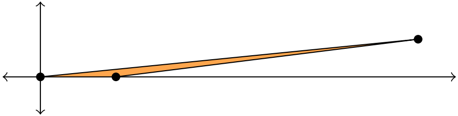

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Even determining if a region contains *any* integer points can be difficult.

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- very hard: we can solve the problem by checking all 2^n assignments of $(X_1, \dots, X_n)$, but we don't even know if there's an algorithm that takes $O(1.999^n)$ steps.
- very important: if we have good heuristics for it, lots of real-life problems become easier to attack.

Boolean satisfiability and integer programming

Encode each Boolean variable X_i by an integer variable x_i with $0 \leq x_i \leq 1$: $X_i =$ **TRUE** corresponds to $x_i = 1$ and $X_i =$ **FALSE** corresponds to $x_i = 0$.

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Then X_1 **OR** X_2 **OR** … **OR** X_k is equivalent to an inequality:

$$x_1 + x_2 + \cdots + x_k \geq 1.$$

We can write $\textbf{NOT}(X_i)$ as $(1 - x_i)$.

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So a system of inequalities can represent a logical expression in "conjunctive normal form": an **AND** of **OR**s.

Fact: all logical expressions can be put in this form. So integer programming can model all Boolean satisfiability problems!

Fixed costs

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Example 1: Fixed costs

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- Cost in the objective function $1000w$.
- We can write other constraints in terms of w when they depend on the existence of a warehouse.

Combining constraints with Boolean variables

Example 2: Conditional constraints

The warehouse can store up to 100 red, yellow, or green bananas—but only if it is rented. Otherwise, it can't store any bananas.

Assume $r, y, g \geq 0$ are the number of bananas stored.

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- The unconditional constraint: $r + y + g \leq 100$.
- The conditional constraint: $r + y + g \leq 100w$.
- This simplifies to the unconditional constraint if $w = 1$, but forces $r = y = g = 0$ if $w = 0$.

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- This simplifies to the unconditional constraint if $w = 1$ (if there is a warehouse), and is effectively not present if $w = 0$ (if there is no warehouse).
- This method does not always work (only if there are practical limits on b) and very large values of the big number make the linear program worse to solve.