Cutting Planes

Math 482, Lecture 34

Misha Lavrov

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Example:

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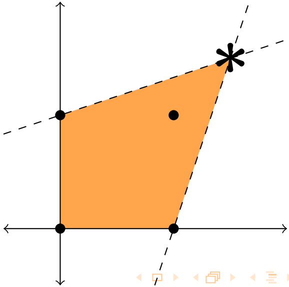

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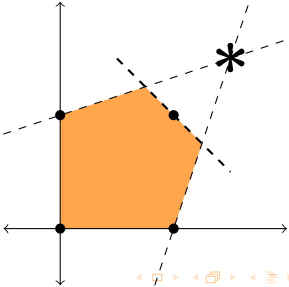

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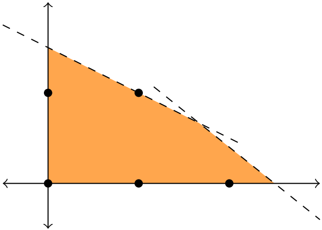
There are lots of methods to generate cutting planes. They vary in quality and in how long they take to find. We'll just talk about one of them.

An example

$$\begin{aligned} \underset{x,y\in\mathbb{Z}}{\text{maximize}} \quad & 2x + 3y \\ \text{subject to} \quad & x + 2y \leq 3 \\ & 4x + 5y \leq 10 \\ & x, y \geq 0 \end{aligned}$$



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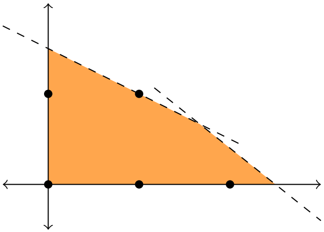

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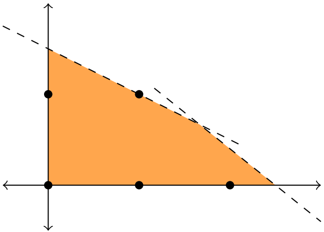

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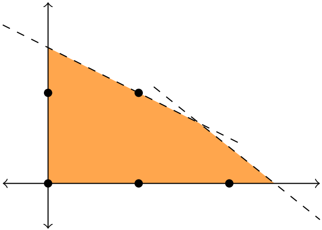

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2. All coefficients in the constraints are integers.

This means that the slacks $s_1 = 3 - (x + 2y)$ and $s_2 = 10 - (4x + 5y)$ are also integers.

Solving the LP relaxation

Starting tableau:

	x	y	s_1	s_2	
s_1	1	2	1	0	3
s_2	4	5	0	1	10
$-z$	2	3	0	0	0

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Pivot on x:

	x	y	s_1	s_2	
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x	1	0	$-5/3$	$2/3$	$5/3$
$-z$	0	0	$-2/3$	$-1/3$	$-16/3$

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An integer that's $\leq \frac{2}{3}$ is ≤ 0, so we can strengthen this:

$$y + s_1 - s_2 \leq 0.$$

This is the Gomory fractional cut.

Alternate form I: solving for x and y

The inequality we get has several equivalent forms. For example,

$$y + s_1 - s_2 \leq 0 \implies y + \Big[3 - (x + 2y)\Big] - \Big[10 - (4x + 5y)\Big] \leq 0$$

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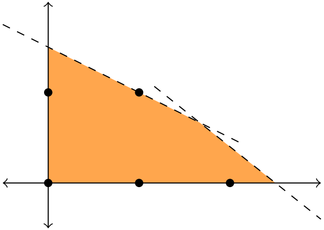

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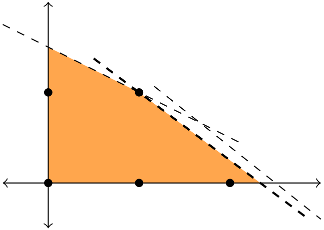

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We can continue with the dual simplex method.

Our new tableau:

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Here, we found the integer optimal solution $(x, y) = (1, 1)$. In general, this may take more cutting plane steps.

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(This last form is the negative of the inequality we added to the tableau.)

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- Add a cutting plane.

How to decide which one to do?

Some LPs are more amenable to cutting planes than others. If we're going to get a really strong cutting plane, we should add it. If it looks like cuts are not working, we can decide to branch.