Approximation Algorithms

Math 482, Lecture 36

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May 4, 2020

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  This is the approach we'll use!

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- Several 2-approximation algorithms for **vertex cover**.

  This means we find a vertex cover whose size is at most twice the optimal size.

- A 2-approximation algorithm for the **traveling salesman problem**.

  This means we find a tour whose cost is at most twice the optimal cost.

The vertex cover problem

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The greedy algorithm is to add nodes to S until we have a vertex cover. If we're clever, we can try to add the node that covers the most still-uncovered edges. This is not going to find the optimal solution, but does it get a good approximation?

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- On one side, a set of vertices A with $|A| = n$.
- On the other side, $k - 1$ sets of vertices $B_2, B_3, \ldots, B_k$ with $|B_i| = \lfloor \frac{n}{i} \rfloor$. (Let $B = B_2 \cup B_3 \cup \cdots \cup B_k$.)

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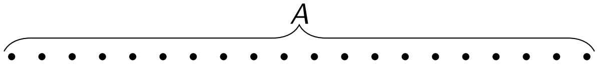
There is a vertex cover of size n: take all of A. But we'll see that the greedy algorithm finds the vertex cover B instead!

The construction

Here is the graph $G_{20,5}$ as an example:



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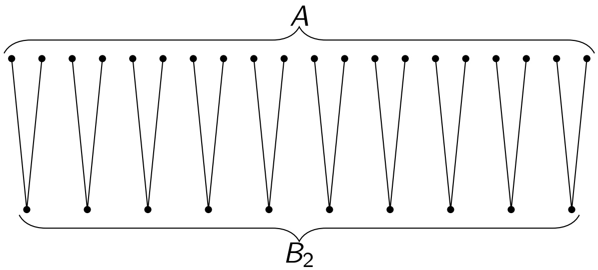

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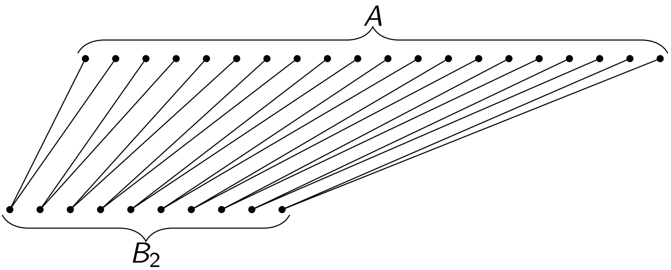

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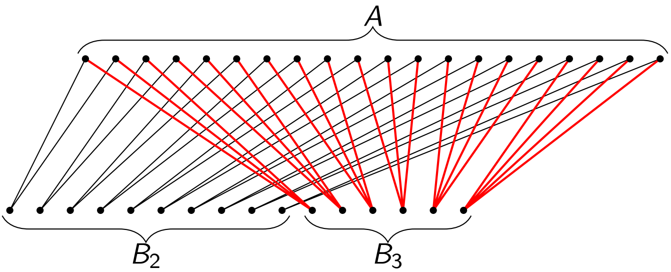

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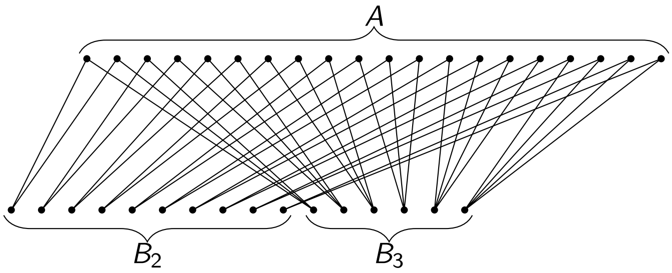

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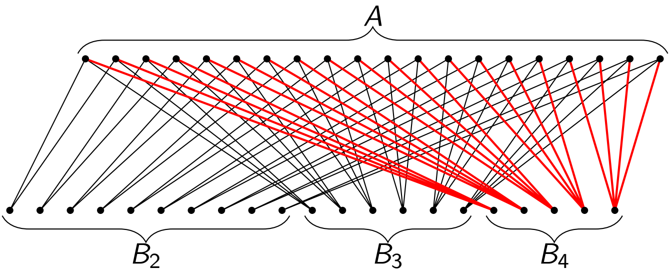

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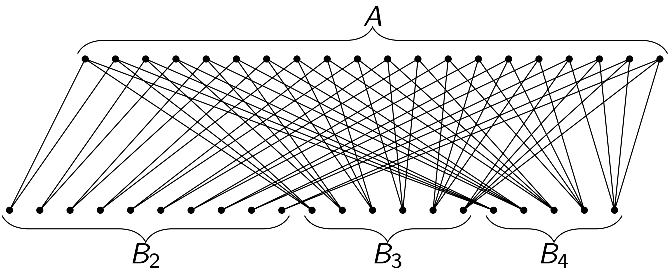

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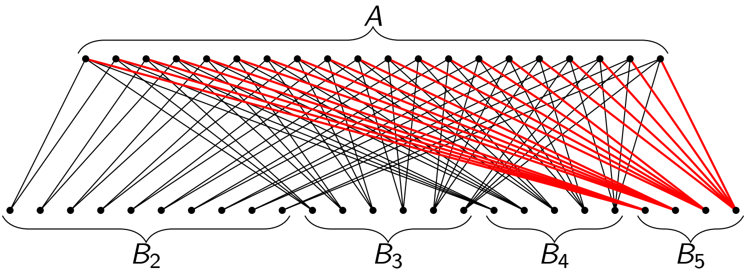

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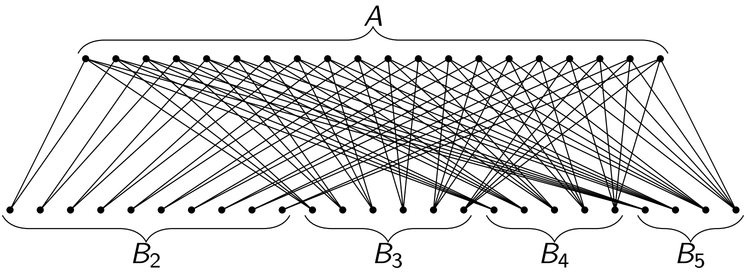

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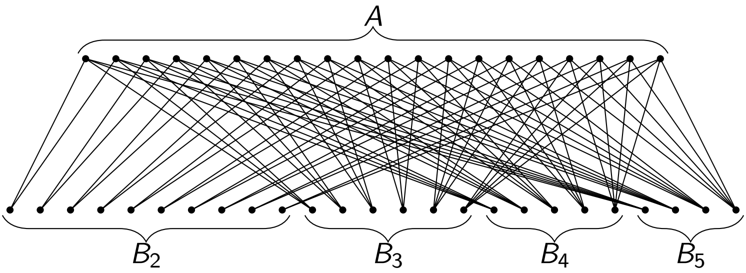

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Each vertex in A has $k-1$ neighbors. But the vertices in B_k have k or $k+1$ neighbors, so the algorithm will choose them first.

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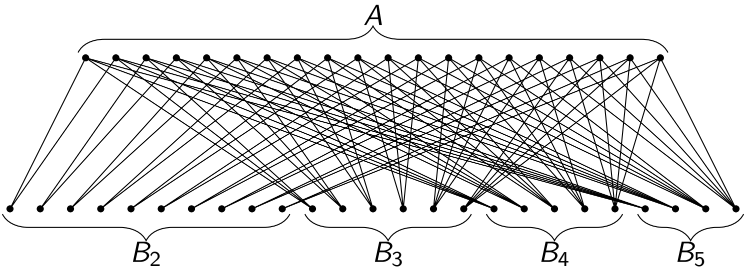

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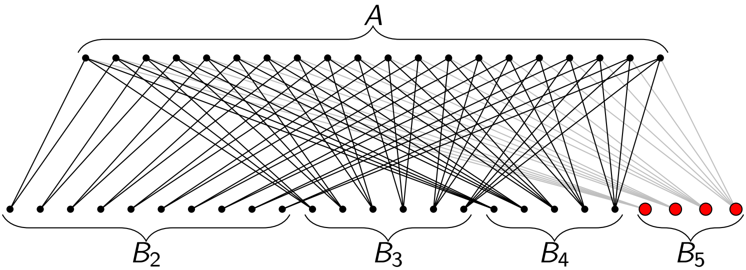

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But now, each vertex in A can cover one fewer uncovered edge. They look worse than vertices in B_{k-1}, so those will be picked next.

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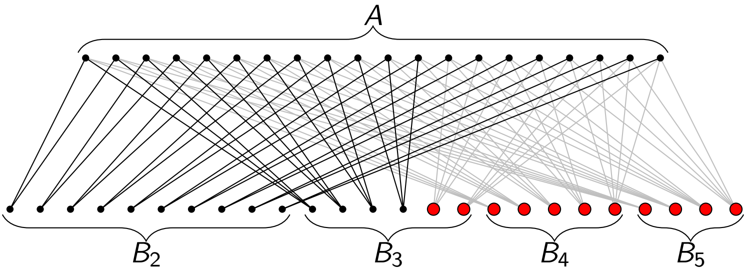

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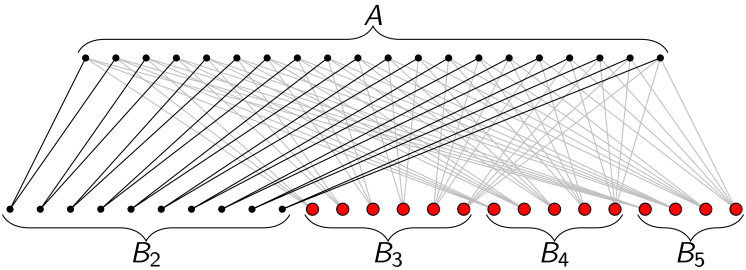

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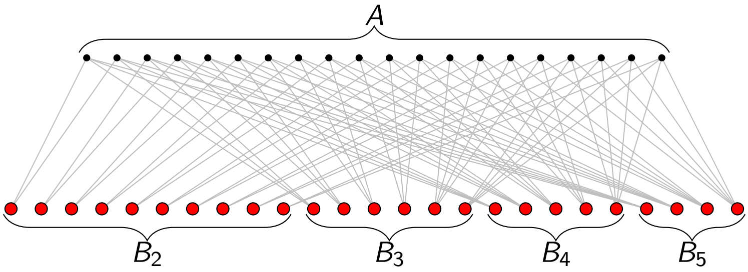

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Eventually, we'll have picked all of B. The approximation ratio is $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$, which can be arbitrarily bad.

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But we've only taken $2|M|$ vertices, so we have a 2-approximation algorithm!

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Our previous 2-approximation algorithm doesn't work anymore: if one endpoint of an edge has weight 1 and the other has weight 99, then choosing both endpoints is 100 times as bad as choosing one endpoint!

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Why is $\mathbf{x}'$ still a feasible solution? Because if $x_i + x_j \geq 1$, then either $x_i \geq \frac{1}{2}$ or $x_j \geq \frac{1}{2}$ (or both), so either $x_i' = 1$ or $x_j' = 1$.

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How good is this approximation algorithm?

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How good is this approximation algorithm? We always have $x_i' \leq 2x_i$, so the weight of $\mathbf{x}'$ is at most twice the weight of $\mathbf{x}$. Since $\mathbf{x}$ is better than the best integer solution, we have a 2-approximation.

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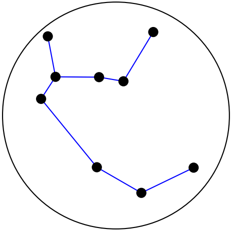
- We'll assume without proving it that it's possible to quickly find a minimum-cost *spanning tree* (it is).

Spanning trees

Suppose we have the min-cost spanning tree.



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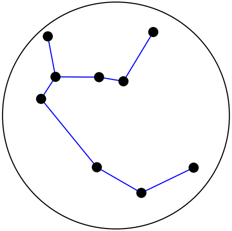

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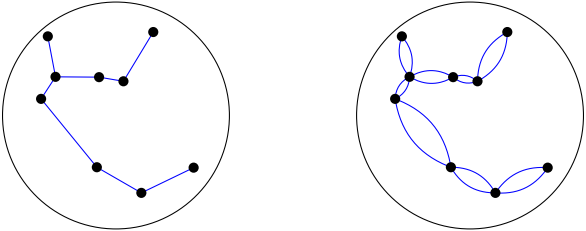

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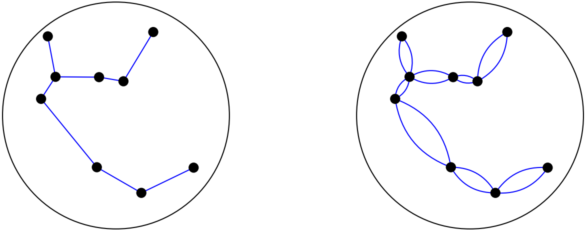

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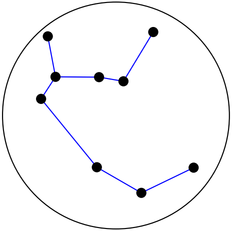

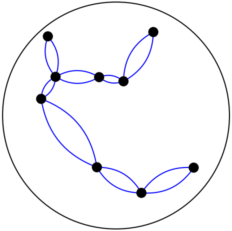

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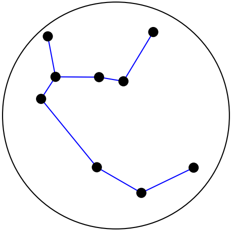

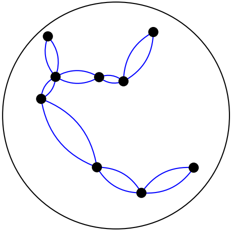

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- So the optimal tour has at least the cost of the min-cost spanning tree.

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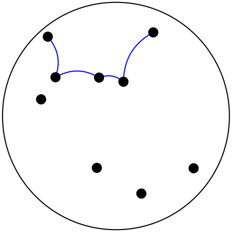

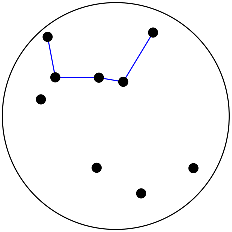

Shortcuts

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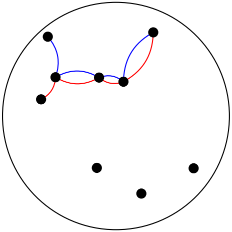

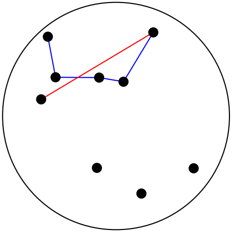

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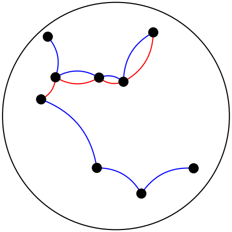

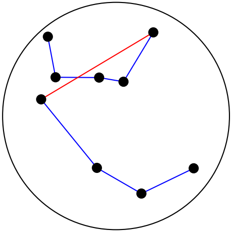

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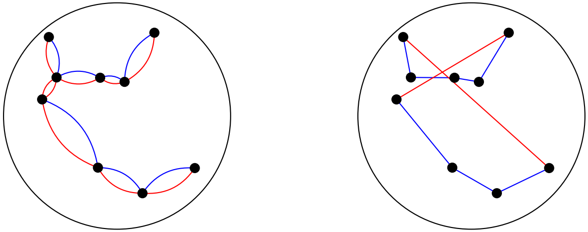

Taking shortcuts only decreases the total cost. So we end up with a tour that's a 2-approximation of the optimal tour.