Chapter 2, Lecture 5: The AM-GM inequality

1 The AM-GM inequality

The AM-GM (Arithmetic Mean - Geometric Mean) inequality states the following:

Theorem 1.1. *For any $x_1, x_2, \dots, x_n \geq 0$,*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$$

with equality only if $x_1 = x_2 = \cdots = x_n$.

The left-hand side of the inequality is the arithmetic mean of the values $x_1, x_2, \dots, x_n$: the usual notion of "average". The right-hand side of the inequality is the ir geometric mean.

The geometric mean is an unusual notion of average that makes sense when we consider values that combine multiplicatively, rather than additively. For example, if an investment grows by 5% for seven months and 2% for five more, we can calculate the average monthly growth by taking the geometric mean of the twelve values

$$1.05, 1.05, 1.05, 1.05, 1.05, 1.05, 1.05, 1.02, 1.02, 1.02, 1.02, 1.02.$$

In this example, we get approximately 1.03739, for an average growth rate of 3.739%. We can interpret this as saying that if the investment grew by 3.739% every month for all twelve months, it would end at the same final value.

The AM-GM inequality also has a weighted form:

Theorem 1.2. *For any $x_1, x_2, \dots, x_n \geq 0$ and for any weights $\delta_1, \delta_2, \dots, \delta_n > 0$ with $\delta_1 + \delta_2 + \cdots + \delta_n = 1$,*

$$\delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_n x_n \geq x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n}$$

with equality only if $x_1 = x_2 = \cdots = x_n$.

We can recover the unweighted AM-GM inequality from its weighted form by setting $\delta_1 = \delta_2 = \cdots = \delta_n = \frac{1}{n}$.

The AM-GM inequality is really just one particular instance Jensen's inequality in disguise. Let $f(t) = -\ln t$: this is a strictly convex function on $(0, \infty)$, since $f''(t) = \frac{1}{t^2} > 0$ for all t. Jensen's inequality says that

$$f(\delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_n x_n) \leq \delta_1 f(x_1) + \delta_2 f(x_2) + \cdots + \delta_n f(x_n).$$

When $x_1, x_2, \dots, x_n$ are not all equal, because f is strictly convex, we get a $>$ in this inequality. That's where the equality condition of AM-GM comes from.

[1] This document comes from the Math 484 course webpage: https://faculty.math.illinois.edu/~mlavrov/courses/484-spring-2019.html

Now let's try to simplify this inequality a bit. Once we replace f by its definition, we get

$$-\ln(\delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_n x_n) \le -\delta_1 \ln x_1 - \delta_2 \ln x_2 - \cdots - \delta_n \ln x_n$$

and we can negate both sides to reverse the inequality:

$$\ln(\delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_n x_n) \ge \delta_1 \ln x_1 + \delta_2 \ln x_2 + \cdots + \delta_n \ln x_n.$$

Now get rid of the ln by applying e^x to both sides:

$$\begin{aligned} \delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_n x_n &\ge e^{\delta_1 \ln x_1 + \delta_2 \ln x_2 + \cdots + \delta_n \ln x_n} \\ &= e^{\delta_1 \ln x_1} e^{\delta_2 \ln x_2} \cdots e^{\delta_n \ln x_n} \\ &= x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n}. \end{aligned}$$

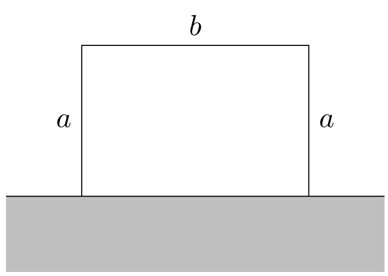
This gives us the weighted AM-GM inequality.

(A minor note: f is convex on $(0, \infty)$ and not even defined at 0, but we stated AM-GM for $x_1, x_2, \ldots, x_n \ge 0$. Is this a problem? It's easily fixed: when $x_i = 0$ for any i, then $x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n}$ immediately becomes 0. On the other side, the arithmetic mean remains nonnegative, and it's strictly positive unless $x_1 = x_2 = \cdots = x_n = 0$. So we're still good.)

2 Applications

2.1 Another traditional calculus problem

Suppose we want to fence off a rectangular region on one side of a very large barn:



If we must fence off at least 50 square meters, how much fencing do we need?

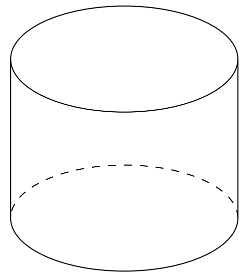
Algebraically, we want to minimize $2a + b$ given that $ab \ge 50$ (and $a, b \ge 0$). By the AM-GM inequality applied to $2a$ and b with equal weights, we have

$$\frac{2a + b}{2} \ge \sqrt{(2a)(b)} \implies \frac{2a + b}{2} \ge \sqrt{2 \cdot 50} \implies 2a + b \ge 20.$$

So we need at least 20 meters of fencing, and we achieve this minimum only when $2a = b$, which means $a = 5$ and $b = 10$.

2.2 A slightly fancier calculus problem

Let's maximize the volume of a cylinder whose surface area is fixed at 200π.



The volume of a cylinder is $\pi r^2 h$ and the surface area is $2\pi r^2 + 2\pi rh$. So we want to maximize $\pi r^2 h$ (just maximizing $r^2 h$ is enough) given that

$$2\pi r^2 + 2\pi rh = 200\pi \iff r^2 + rh = 100.$$

Here's something that doesn't work: applying the AM-GM inequality to r^2 and rh. Then we have

$$\frac{r^2 + rh}{2} \geq \sqrt{r^2 \cdot rh} \iff r^3 h \leq \left(\frac{100}{2}\right)^2 = 2500$$

which would be fine if we were trying to maximize $r^3 h$, but we want to maximize $r^2 h$ instead.

Instead, we apply the weighted AM-GM inequality, weighting $3r^2$ at $\frac{1}{3}$ and $\frac{3}{2}rh$ at $\frac{2}{3}$. We have

$$\frac{1}{3}(3r^2) + \frac{2}{3}(\tfrac{3}{2}rh) \geq (3r^2)^{1/3}(\tfrac{3}{2}rh)^{2/3}.$$

The left-hand side simplifies to $r^2 + rh$ again, which we know is 100. The right-hand side can be rewritten as $3 \cdot 2^{-2/3} \cdot (r^2 h)^{2/3}$.

So we get

$$3 \cdot 2^{-2/3} \cdot (r^2 h)^{2/3} \leq 100 \implies \frac{3^{3/2}}{2} \cdot r^2 h \leq 100^{3/2} = 1000 \implies \pi r^2 h \leq \frac{2000\pi}{3\sqrt{3}}.$$

This is an upper bound on $\pi r^2 h$, but we know that we can achieve equality when the two things we're averaging in the AM-GM inequality are equal. That is, equality occurs when $3r^2 = \frac{3}{2}rh$, or $r = \frac{1}{2}h$.

(More precisely, we want $3r^2 = \frac{3}{2}rh = 100$, so $r = \sqrt{\frac{100}{3}} = \frac{10}{\sqrt{3}}$, and $h = 2r = \frac{20}{\sqrt{3}}$.)

You may, by now, be feeling some sort of niggling doubts: how did we know to weight $3r^2$ at $\frac{1}{3}$ and $\frac{3}{2}rh$ at $\frac{2}{3}$, rather than (for example) weight $5r^2$ at $\frac{1}{5}$ and $\frac{5}{4}rh$ at $\frac{4}{5}$?

2.3 An unconstrained example

Here's a more abstract example, which represents the kind of problem we'll solve more generally in the next few lectures.

The problem is this: for $x, y > 0$, minimize $f(x,y) = 2xy + \frac{y}{x^2} + \frac{3x}{y}$.

Once again, there is a "magic" solution, which involves pulling weights for the AM-GM inequality out of the blue sky. We have

$$\begin{aligned} f(x,y) = 2xy + \frac{y}{x^2} + \frac{3x}{y} &= \frac{1}{6}(12xy) + \frac{1}{3}(3x^{-2}y) + \frac{1}{2}(6xy^{-1}) \\ &\geq (12xy)^{1/6}(3x^{-2}y)^{1/3}(6xy^{-1})^{1/2} \\ &= 12^{1/6}3^{1/3}6^{1/2}x^{1/6}y^{1/6}x^{-2/3}y^{1/3}x^{1/2}y^{-1/2}. \end{aligned}$$

This looks like a mess, but—miraculously—the powers of x and y all cancel, giving us

$$f(x,y) \geq 12^{1/6} \cdot 3^{1/3} \cdot 6^{1/2} = 2^{5/6} \cdot 3.$$

Moreover, we could find values of x and y that give us this value of f, by setting $12xy = 3x^{-2}y = 6xy^{-1}$, which produces $x = 2^{-2/3}$ and $y = 2^{-1/2}$.

In preparation for the next lecture, here is something to think about. How did we come up with the coefficients $\delta_1 = \frac{1}{6}$, $\delta_2 = \frac{1}{3}$, and $\delta_3 = \frac{1}{2}$?

There are some constraints on the coefficients: we want them to be positive and add up to 1, because the AM-GM inequality requires that, and we want the powers of x and y we get to go poof like they did in the calculation above. You should think about what that actually cashes out to as a constraint on δ_1, δ_2, and δ_3.