Math 484: Nonlinear Programming[1] Mikhail Lavrov

Chapter 5, Lecture 9: The Geometric Programming Dual

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1 The geometric programming dual, in general

In general, a constrained geometric program has positive variables $t_1, t_2, \dots, t_m$. It has the form

$$(GP) \qquad \begin{cases} \underset{\mathbf{t}>\mathbf{0}}{\text{minimize}} & \text{Term}_1(\mathbf{t}) + \cdots + \text{Term}_{n_1}(\mathbf{t}) \\ \text{subject to} & \text{Term}_{n_1+1}(\mathbf{t}) + \cdots + \text{Term}_{n_2}(\mathbf{t}) \le 1, \\ & \text{Term}_{n_2+1}(\mathbf{t}) + \cdots + \text{Term}_{n_3}(\mathbf{t}) \le 1, \\ & \dots \\ & \text{Term}_{n_{k-1}+1}(\mathbf{t}) + \cdots + \text{Term}_{n_k}(\mathbf{t}) \le 1. \end{cases}$$

Each term $\text{Term}_i(\mathbf{t}) = C_i t_1^{\alpha_{i1}} t_2^{\alpha_{i2}} \cdots t_m^{\alpha_{im}}$ is a posynomial term : $C_i > 0$ and $\alpha_{i1}, \dots, \alpha_{im}$ are arbitrary real numbers. For each of the terms, whether it appeared in the objective function or in a constraint, we have a dual variable δ_i.

The dual objective function $v(\boldsymbol{\delta})$ is the product of:

- A $\left(\frac{C_i}{\delta_i}\right)^{\delta_i}$ factor for each dual variable.
- For each constraint, we have a special factor:

  $$\left(\delta_{n_i+1} + \delta_{n_i+2} + \cdots + \delta_{n_{i+1}}\right)^{\delta_{n_i+1}+\delta_{n_i+2}+\cdots+\delta_{n_{i+1}}}.$$

  The variables that appear in this factor correspond to the terms that appear in that constraint.

The dual problem has the following constraints:

- For each primal variable t_j, we get a constraint

  $$\delta_1\alpha_{11} + \delta_2\alpha_{21} + \cdots + \delta_n\alpha_{n1} = 0$$

  where the coefficient of δ_i is the power of t_j in the i^{th} term $\text{Term}_i(\mathbf{t})$.
- There is a normalization constraint $\delta_1 + \delta_2 + \cdots + \delta_{n_1} = 1$, where $\delta_1, \delta_2, \dots, \delta_{n_1}$ are the dual variables corresponding to the terms in the primal objective function.
- There is a positivity constraint $\boldsymbol{\delta} > \mathbf{0}$. It has an exception: for each constraint, we are allowed to set *all* dual variables from that constraint to 0 simultaneously. (For the purposes of evaluating $v(\boldsymbol{\delta})$, we assume that $0^0 = 1$ in this case.)

[1]This document comes from the Math 484 course webpage: https://faculty.math.illinois.edu/~mlavrov/courses/484-spring-2019.html

1.1 Wait... positivity constraint?

Okay, when we derived the constraints on the geometric programming dual last time, we did not have any kind of requirement that $\boldsymbol{\delta} > \mathbf{0}$. We just had $\boldsymbol{\delta} \geq \mathbf{0}$, because we started from $\boldsymbol{\lambda} \geq \mathbf{0}$, which is always there in the KKT dual.

In deriving the dual program, we set the i^{th} term of the geometric program equal to e^{z_i}, and the Lagrangian contained the expression $e^{z_i} - \lambda_i z_i$ (for a term in the objective function) or the expression $\mu_j e^{z_i} - \lambda_i z_i$ (for a term in the j^{th} constraint). This is minimized (as a function of z_i) when $z_i = \log \lambda_i$, or when $z_i = \log \frac{\lambda_i}{\mu_j}$, respectively.

This doesn't work when $\lambda_i = 0$. In that case, e^{z_i} or $\mu_k e^{z_i}$ is minimized by taking $z_i \to -\infty$. This happens when we'd like to set $\text{Term}_i(\mathbf{t}) = 0$, but we can only make it arbitrarily small; for example, if you're minimizing $1 + \frac{1}{x}$, you'd like to take $x \to \infty$ to get as close to 1 as possible.

But this does not actually correspond to a feasible primal solution, and so we forbid this from happening. Instead, we require $\boldsymbol{\delta} > 0$, to limit ourselves only to cases where the primal program will have an optimal solution.

There is an exception to the exception. Suppose that μ_j, the dual variable corresponding to the constraint $e^{z_{n_j+1}} + \cdots + e^{z_{n_{j+1}}} \leq 1$, is 0. In this case, the expression $\mu_j e^{z_i} - \lambda_i z_i$ simply becomes $-\lambda_i z_i$, and we *must* set $\lambda_i = 0$ to make $h(\boldsymbol{\mu}, \boldsymbol{\lambda}) > -\infty$.

This gives us a weird positivity constraint. The dual variables corresponding to the terms in the objective function *must* be always positive. The other dual variables have an escape clause: they are usually positive, but we can set some of them to 0, as long as *all* or *none* of the dual variables from any given primal constraint are 0. Intuitively, this corresponds to the case where a constraint is unnecessary.

2 Using a dual solution to find a primal solution

Once the optimal dual solution $\boldsymbol{\delta}^*$ is found, we can use it to find an optimal primal solution $\mathbf{t}^*$. To do so, we use the following equations; essentially, we know the values of many of the *terms* in the primal program.

2.1 Terms appearing in the objective function

As before, with the unconstrained geometric program, when $\boldsymbol{\delta}^*$ is an optimal dual solution, the optimal primal solution is found by solving:

$$\text{Term}_1(\mathbf{t}^*) = \delta_1^* v(\boldsymbol{\delta}^*), \quad \text{Term}_2(\mathbf{t}^*) = \delta_2^* v(\boldsymbol{\delta}^*), \quad \ldots, \quad \text{Term}_{n_1}(\mathbf{t}^*) = \delta_{n_1}^* v(\boldsymbol{\delta}^*).$$

Where does this come from in the KKT dual?

We have $z_i = \log \lambda_i$, or $\text{Term}_i(\mathbf{t}^*) = e^{z_i} = \lambda_i$. But we don't have access to λ_i directly: we just have the normalized variable δ_i. So by default, we just know that the proportions

$$\text{Term}_1(\mathbf{t}^*) : \text{Term}_2(\mathbf{t}^*) : \cdots : \text{Term}_{n_1}(\mathbf{t}^*) \qquad \text{and} \qquad \delta_1 : \delta_2 : \cdots : \delta_{n_1}$$

are equal.

But recall that when the primal and dual have an optimal solution, their objective values are equal. The dual objective value is $v(\boldsymbol{\delta}^*)$. So we must have

$$\text{Term}_1(\mathbf{t}^*) + \text{Term}_2(\mathbf{t}^*) + \cdots + \text{Term}_{n_1}(\mathbf{t}^*) = v(\boldsymbol{\delta}^*)$$

and this, together with the ratios between the terms, tells us their values.

2.2 Terms appearing in active constraints

Every primal constraint whose dual variables are positive is an *active* constraint: the value of the left-hand side is not just at most 1 but equal to 1. For all such constraints, we have

$$\text{Term}_{n_i+1}(\mathbf{t}^*) = \frac{\delta^*_{n_i+1}}{\delta_{n_i+1} * + \cdots + \delta^*_{n_{i+1}}}, \quad \ldots, \quad \text{Term}_{n_{i+1}}(\mathbf{t}^*) = \frac{\delta^*_{n_{i+1}}}{\delta_{n_i+1} * + \cdots + \delta^*_{n_{i+1}}}.$$

Where does this come from in the KKT dual?

We have $z_i = \log \frac{\lambda_i}{\mu_j}$ for such a term, or $\text{Term}_i(\mathbf{t}^*) = e^{z_i} = \frac{\lambda_i}{\mu_j}$. Again, we don't have access to the $\boldsymbol{\lambda}$ vector directly, just its normalized version $\boldsymbol{\delta}$. So by default, all we can say is that the proportions

$$\text{Term}_{n_i+1}(\mathbf{t}^*) : \text{Term}_{n_i+2}(\mathbf{t}^*) : \cdots : \text{Term}_{n_{i+1}}(\mathbf{t}^*) \qquad \text{and} \qquad \delta_{n_i+1} : \delta_{n_i+2} : \cdots : \delta_{n_{i+1}}$$

are equal.

But for an active constraint, the sum

$$\text{Term}_{n_i+1}(\mathbf{t}^*) + \text{Term}_{n_i+2}(\mathbf{t}^*) + \cdots + \text{Term}_{n_{i+1}}(\mathbf{t}^*)$$

must be equal to 1. So using this, and the ratio between the terms, we can find out what the values of the terms are.

3 Example

The geometric program

$$(GP) \qquad \begin{cases} \underset{x,y,z>0}{\text{minimize}} & \dfrac{1}{xyz} \\ \text{subject to} & x + y \le 1, \\ & y + z \le 1 \end{cases}$$

has dual

$$(D) \qquad \begin{cases} \underset{\boldsymbol{\delta} \in \mathbb{R}^5}{\text{maximize}} & \left(\dfrac{1}{\delta_1}\right)^{\delta_1} \left(\dfrac{1}{\delta_2}\right)^{\delta_2} \left(\dfrac{1}{\delta_3}\right)^{\delta_3} \left(\dfrac{1}{\delta_4}\right)^{\delta_4} \left(\dfrac{1}{\delta_5}\right)^{\delta_5} (\delta_2 + \delta_3)^{\delta_2+\delta_3} (\delta_4 + \delta_5)^{\delta_4+\delta_5} \\ \text{subject to} & -\delta_1 + \delta_2 = 0 \\ & -\delta_1 + \delta_3 + \delta_4 = 0 \\ & -\delta_2 + \delta_5 = 0 \\ & \delta_1 = 1 \\ & \boldsymbol{\delta} > \mathbf{0} \text{ with exceptions } \delta_2 = \delta_3 = 0 \text{ and } \delta_4 = \delta_5 = 0. \end{cases}$$

From $\delta_1 = 1$, we deduce that $\delta_2 = \delta_5 = 1$, and $\delta_3 + \delta_4 = 1$. Solving for δ_3 and δ_4 would require evaluating the objective function

$$v(1, 1, \delta_3, \delta_4, 1) = \left(\frac{1}{\delta_3}\right)\left(\frac{1}{\delta_4}\right)(1+\delta_3)^{1+\delta_3}(1+\delta_4)^{1+\delta_4}$$

and trying to maximize it. But, intuitively, we want $\delta_3 = \delta_4$ by symmetry, and so the optimal dual solution is $\boldsymbol{\delta} = (1, 1, \frac{1}{2}, \frac{1}{2}, 1)$.

In theory, we can compute $v(\boldsymbol{\delta}) = \frac{27}{4}$, and deduce that $\frac{1}{xyz} = \frac{27}{4}$ as well. If we're lazy, we can skip this step, because computing $v(\boldsymbol{\delta})$ is painful, and we have many active constraints to choose from.

Since δ_2, δ_3 are nonzero, we have

$$\text{Term}_2(x, y, z) = \frac{\delta_2}{\delta_2 + \delta_3}, \text{Term}_3(x, y, z) = \frac{\delta_3}{\delta_2 + \delta_3}$$

and so $x = \frac{2}{3}$, $y = \frac{1}{3}$.

Similarly, since δ_4, δ_5 are nonzero, we have

$$\text{Term}_4(x, y, z) = \frac{\delta_4}{\delta_4 + \delta_5}, \text{Term}_5(x, y, z) = \frac{\delta_5}{\delta_4 + \delta_5}$$

and so $y = \frac{1}{3}$, $z = \frac{2}{3}$.

This tells us the primal optimal solution.

(It's also a confirmation that the choice $\delta_3 = \delta_4 = \frac{1}{2}$ was correct: if we chose anything else, these two steps would have given us different values for y.)