Math 484: Nonlinear Programming	Mikhail Lavrov

Homework #6

Spring 2019 — *Due Friday, March 15*

1. Solve the optimization problem

$$\begin{aligned} \underset{x,y,z\in\mathbb{R}}{\text{minimize}} \quad & x^2 - xy + y^2 + z^2 \\ \text{subject to} \quad & 3x + z = 13, \\ & 3x - 3y + 2z = 8. \end{aligned}$$

2. Let C be the convex set $\{(x, y) \in \mathbb{R}^2 : x^2 + y \le 2 \text{ and } x + y^2 \le 2\}$.

   Using the fact that $(1, 1)$ is the point of C closest to $(3, 4)$, find a linear inequality which is true for every point of C, false for $(3, 4)$, and holds with equality at $(1, 1)$.

3. Let C be the convex set $\{(x, y) \in \mathbb{R}^2 : x^2 + y \le 2 \text{ and } x + y^2 \le 2\}$.

   Using the fact that $(1, 1)$ is the point of C closest to $(3, 4)$ *and* that $(1, 1)$ is the point of C closest to $(5, 3)$, deduce that $(1, 1)$ is also the point of C closest to $(7, 6)$.

4. For each of the following sets, determine their interior, boundary, and closure. You do not have to justify your answer.

   (a) $S_1 = [1, 2) \cup (3, 4) \cup (4, \infty) \subseteq \mathbb{R}$.

   (b) $S_2 = \{(x, y) \in \mathbb{R}^2 : y = x \text{ and } y \ne 0\}$.

   (c) $S_3 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\} - \{(0, 0, 0)\}$.

5. *(For 4-credit students only. There are no actual students enrolled in this course for 4 credits this semester, but you should think about this problem a little if you have the time.)*

   Let A be a positive definite $n \times n$ matrix, and consider the function $f(\mathbf{x}) = \mathbf{x}^\mathsf{T} A\mathbf{x}$.

   Show that $\frac{f(\mathbf{x})}{\|\mathbf{x}\|^2}$ has a global minimizer on $\mathbb{R}^n - \{\mathbf{0}\}$. *(Hint: the extreme value theorem doesn't apply directly to $\mathbb{R}^n - \{\mathbf{0}\}$, which is neither closed nor bounded, but can you make it apply?)*

   Here's (one reason) why this is useful. Let ϵ be the minimum value of $\frac{f(\mathbf{x})}{\|\mathbf{x}\|^2}$; we must have $\epsilon > 0$ because A is positive definite so both parts of the fraction are positive. Then we have $f(\mathbf{x}) \ge \epsilon\|\mathbf{x}\|^2$ for all $\mathbf{x}$, which can be used to show that $f(\mathbf{x})$ is coercive.