How a quadratic equation solves peg solitaire

Misha Lavrov

April 22, 2023
Kennesaw Math Competition Awards Ceremony

Peg solitaire

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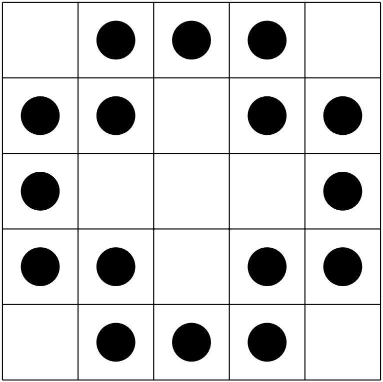

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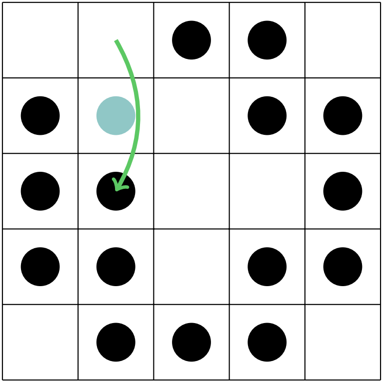

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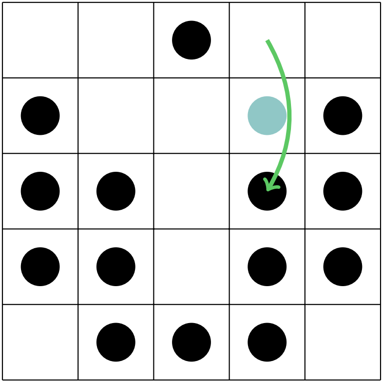

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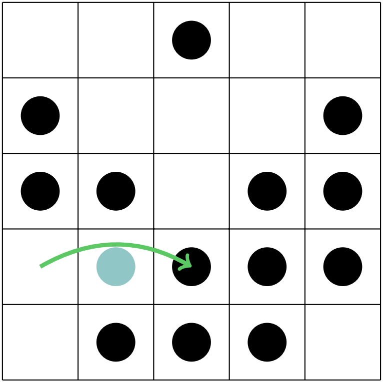

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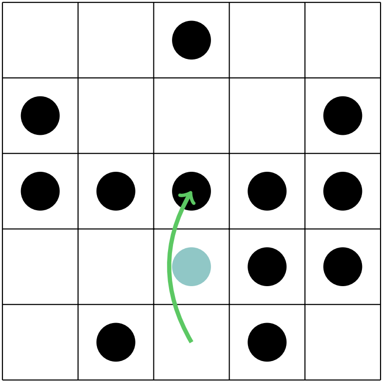

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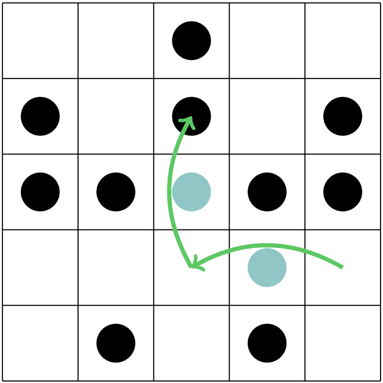

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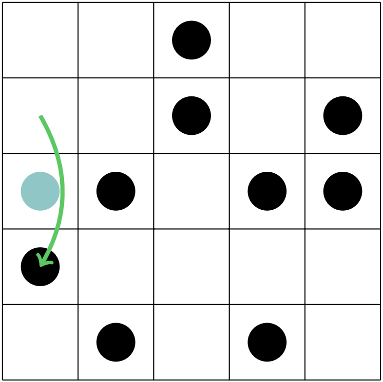

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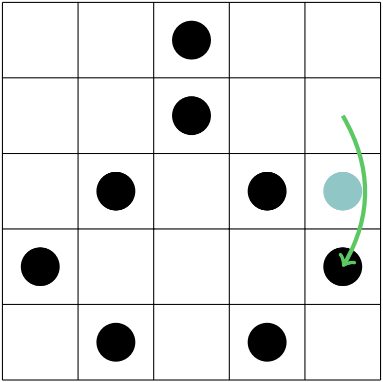

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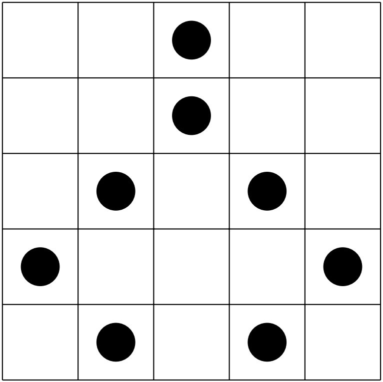

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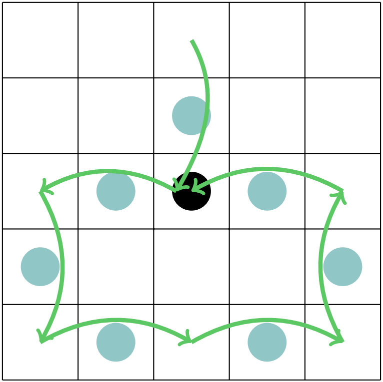

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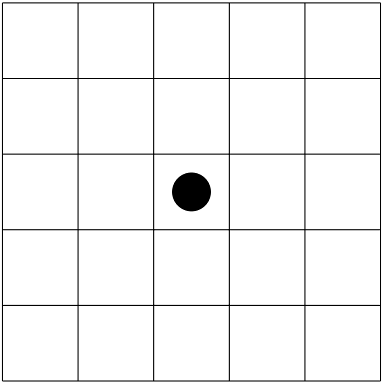

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Conway's soldiers

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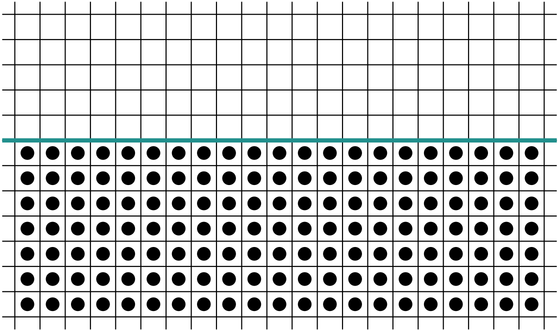
The setup is an infinite board with pegs everywhere **below a certain line**:



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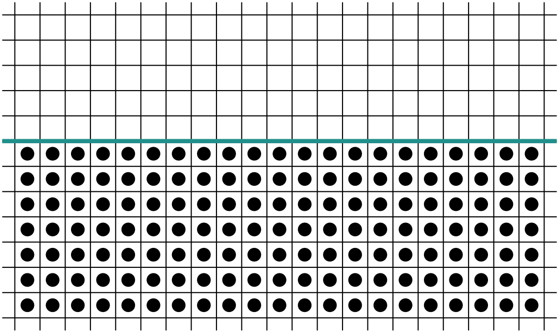

The goal: to get a peg as far above the line as possible.

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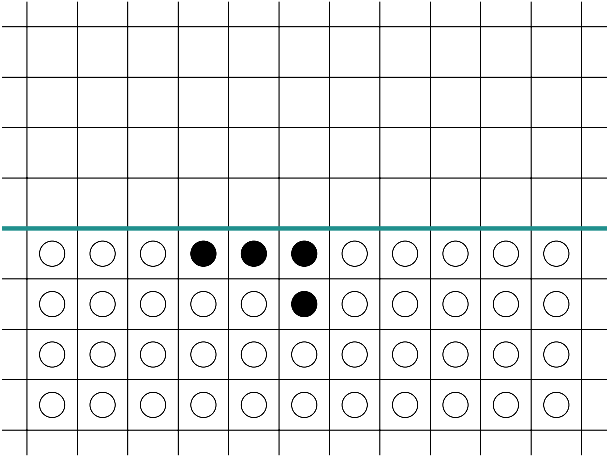

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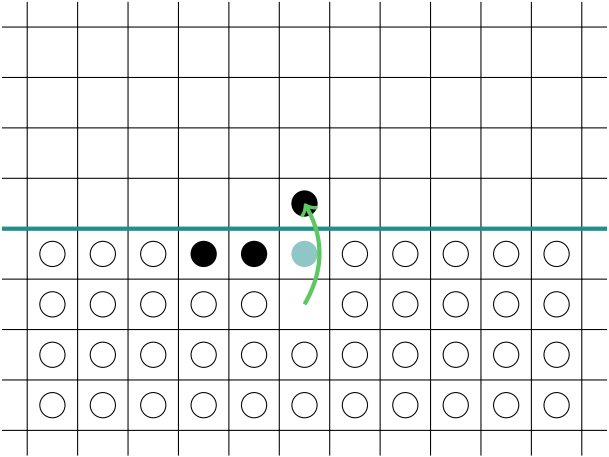

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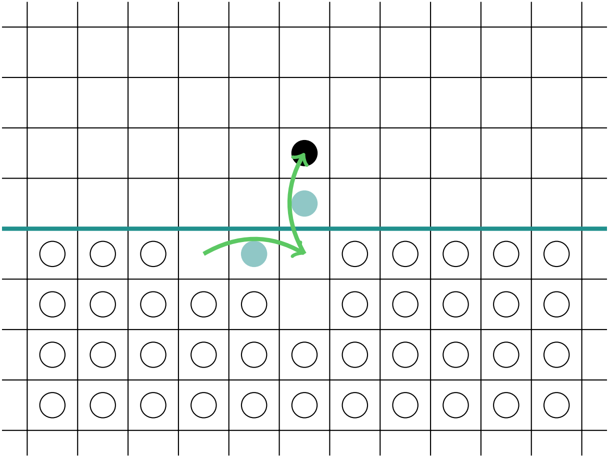

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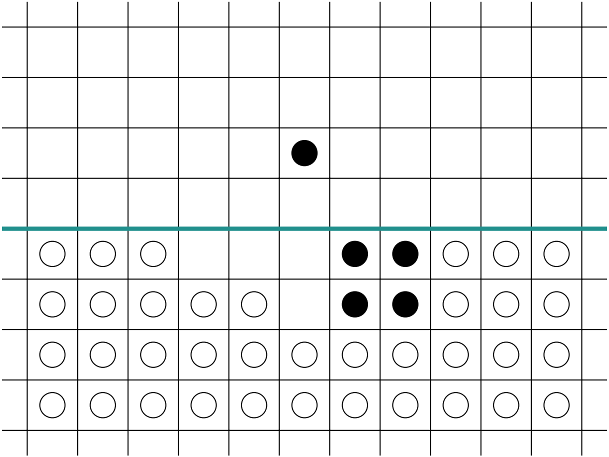

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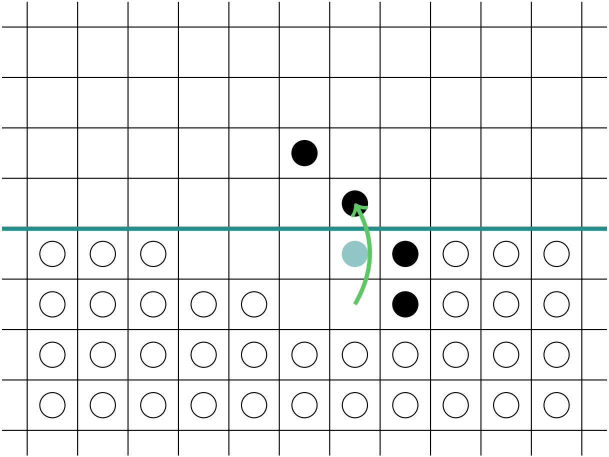

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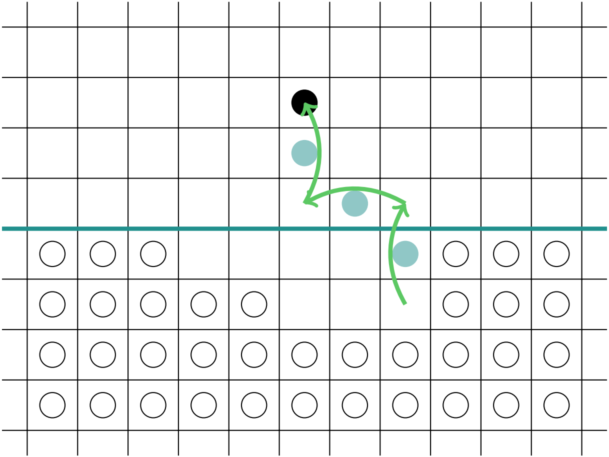
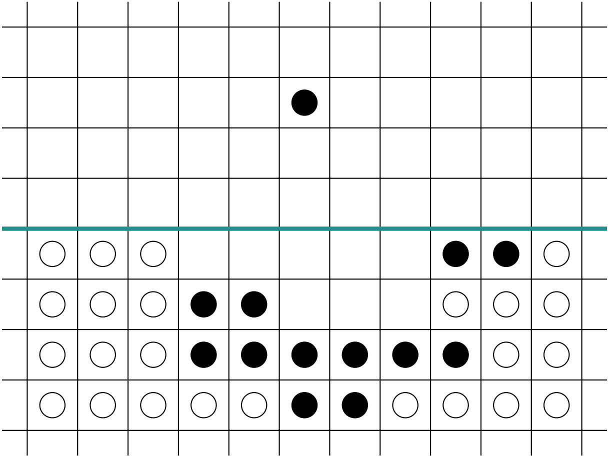


How high can we go? Row 2! Row 3!



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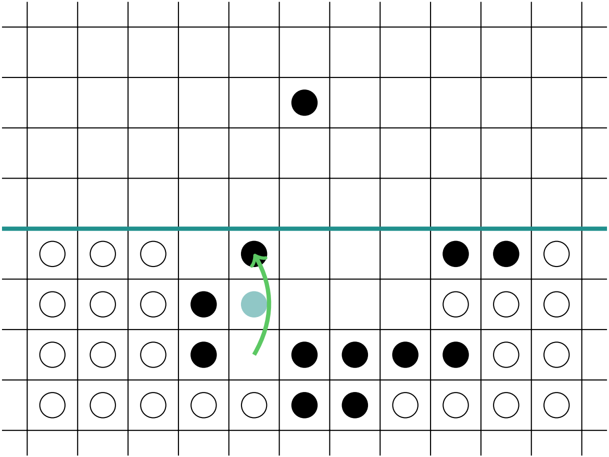

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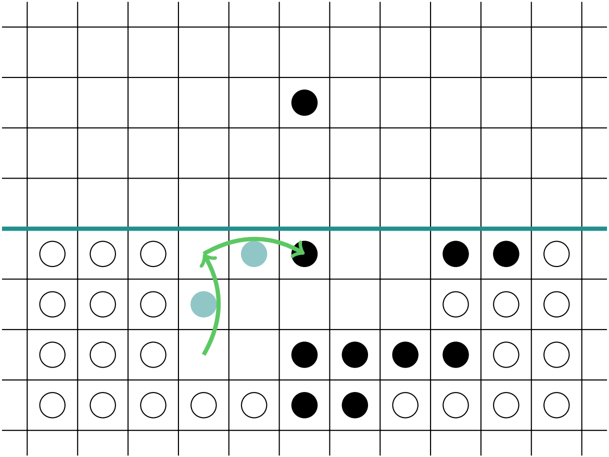

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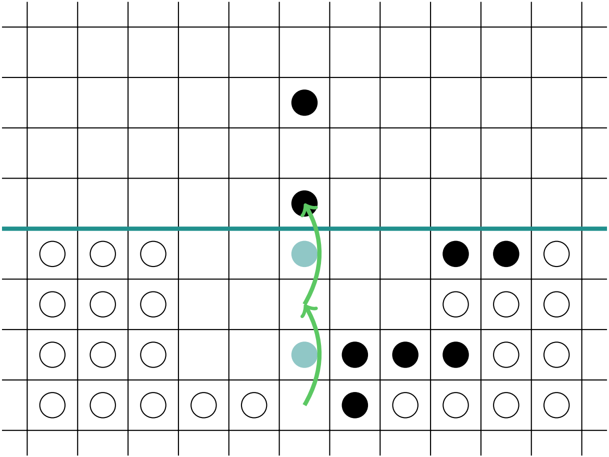

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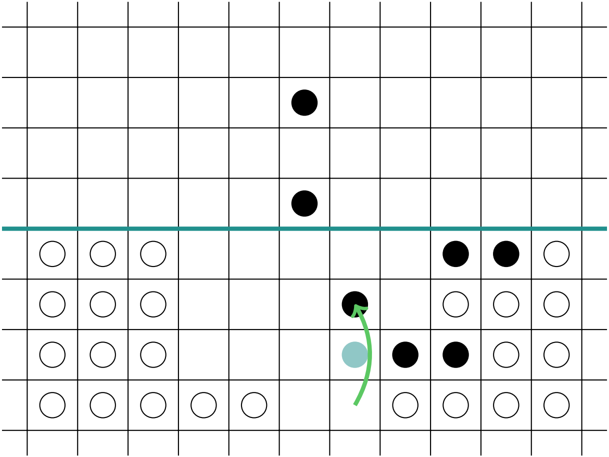

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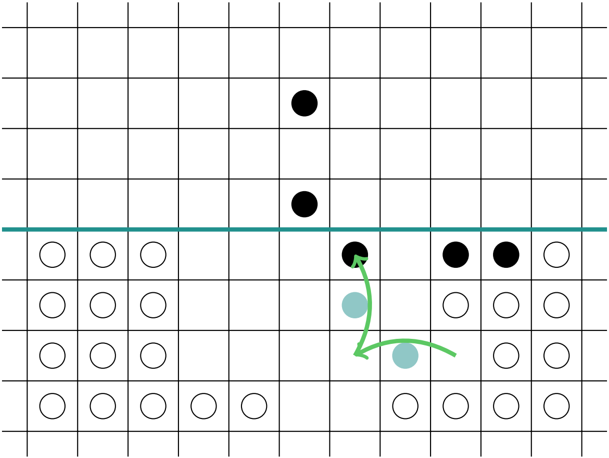
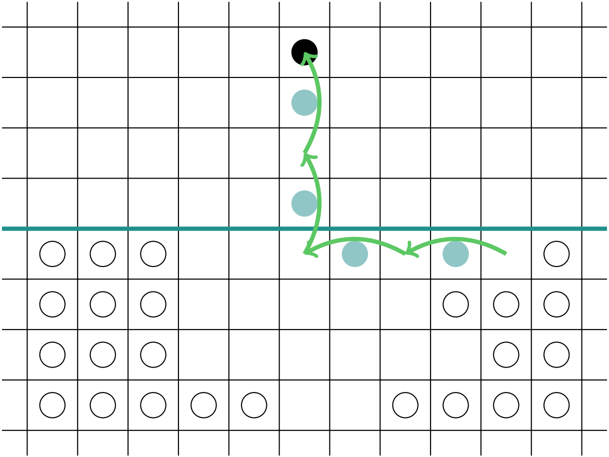


How high can we go? Row 2! Row 3! Row 4!



Warm-up: how many jumps?

Theorem. We need at least 7 jumps to get to row 3.

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Proof. We can assign "point values" based on how close our pegs get to a target in row 3:

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			3	5	8	5	3			
1	1	1	2	3	5	3	2	1	1	1
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So we need to use pegs below the line with a total point value of at least 21. This takes at least 8 pegs: $5+3+3+3+2+2+2+1$.

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The number $\frac{1+\sqrt{5}}{2} \approx 1.618$ is called the **Golden ratio**. It shows up in many places in nature, art, and peg solitaire games.

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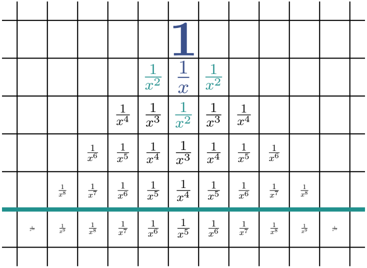

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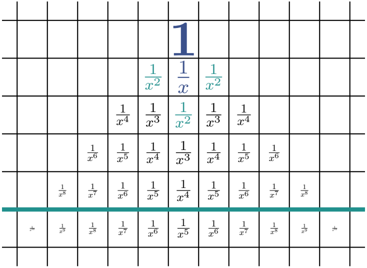

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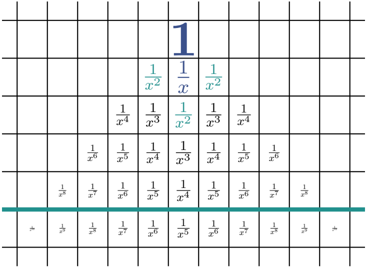

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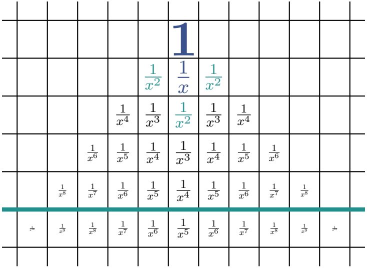

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So no *finite* number of pegs can reach the target!

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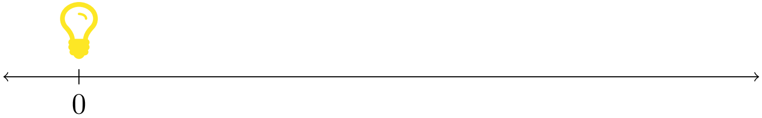

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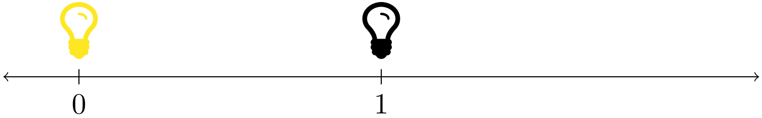

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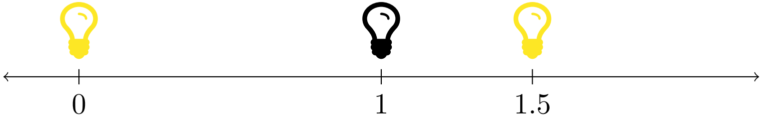

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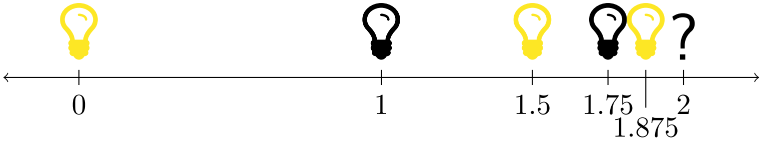

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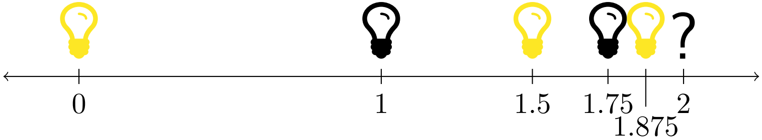

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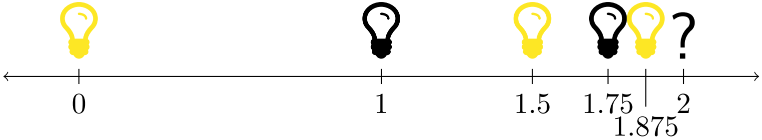

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Negatively inspired by this, we will add a rule: **each cell of our grid should only change state finitely many times**.

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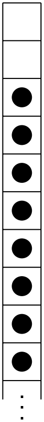

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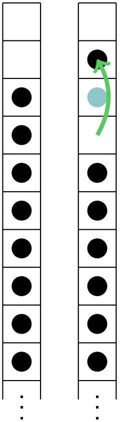

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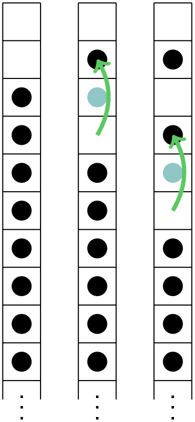

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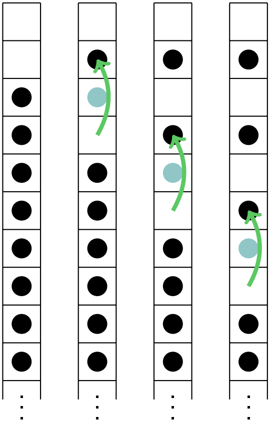

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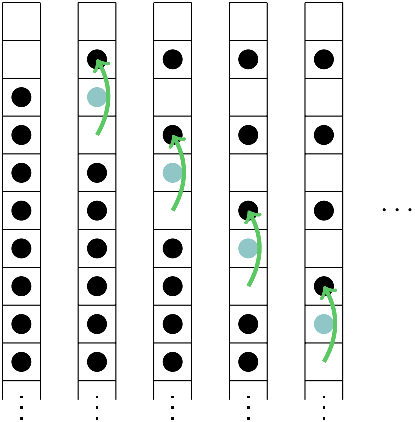

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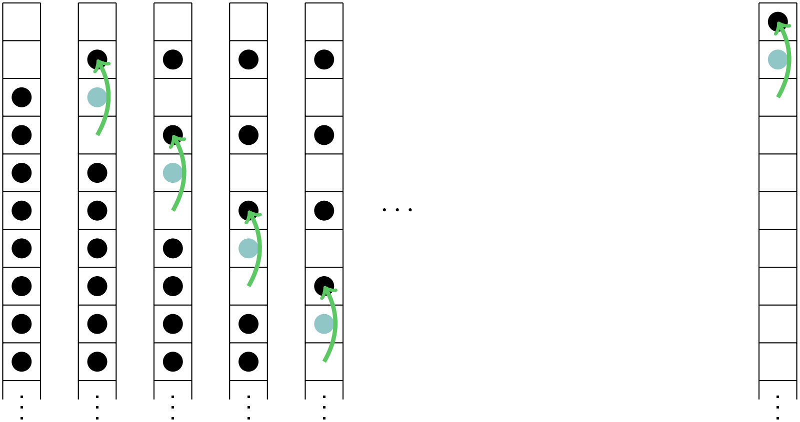

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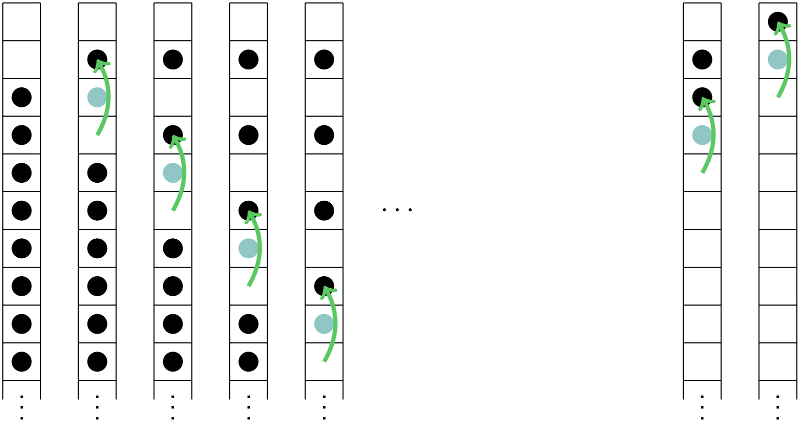

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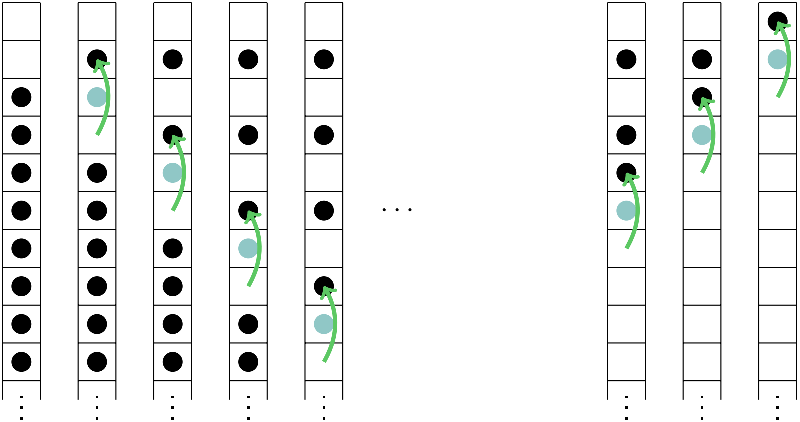

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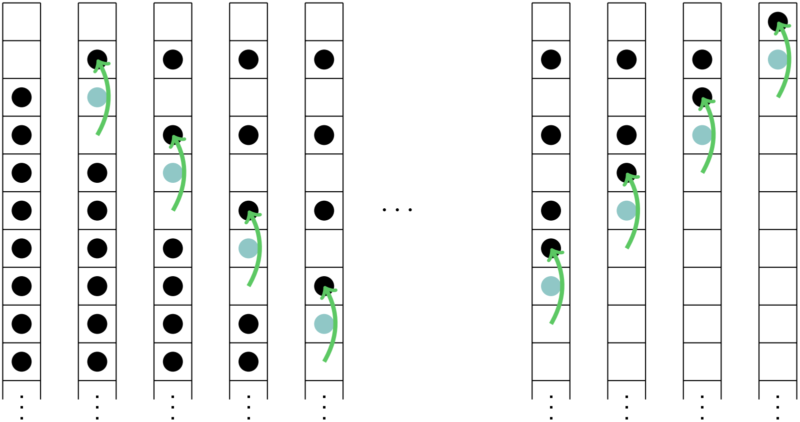

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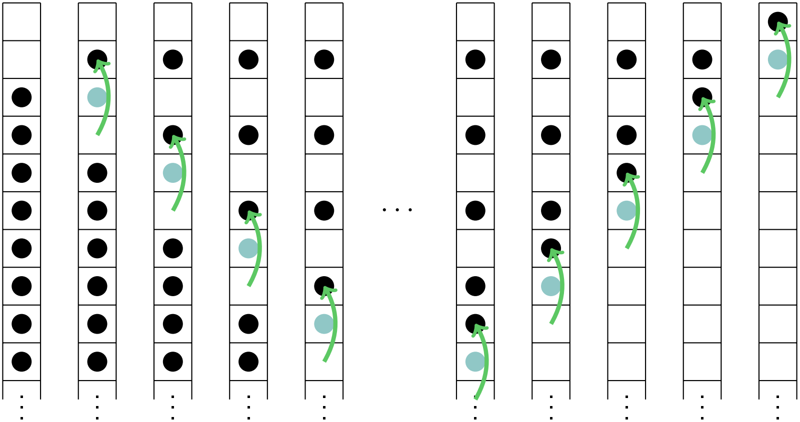

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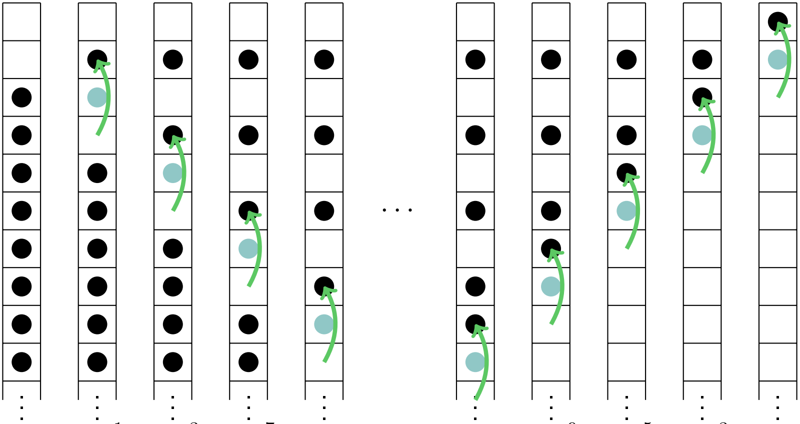
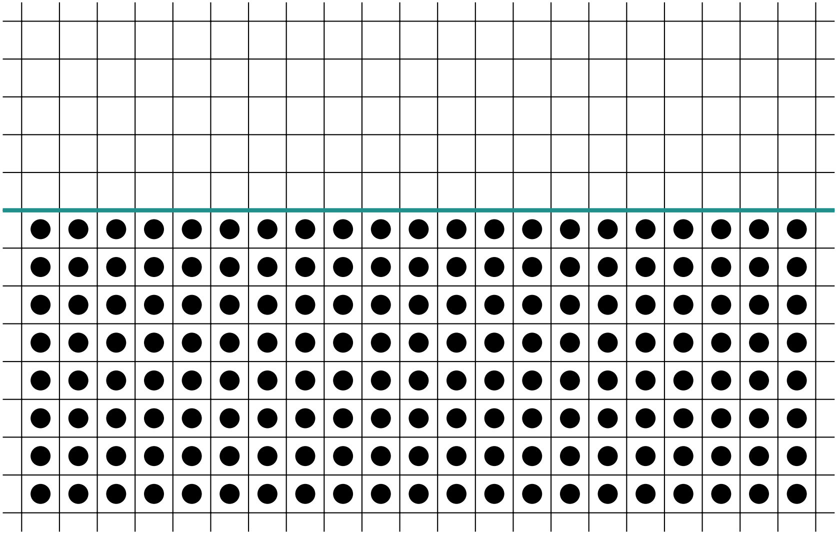


$t = 0$ $t = \frac{1}{2}$ $t = \frac{3}{4}$ $t = \frac{7}{8}$ $t = \frac{15}{16}$ $t = \frac{17}{16}$ $t = \frac{9}{8}$ $t = \frac{5}{4}$ $t = \frac{3}{2}$ $t = 2$

The first half of infinity



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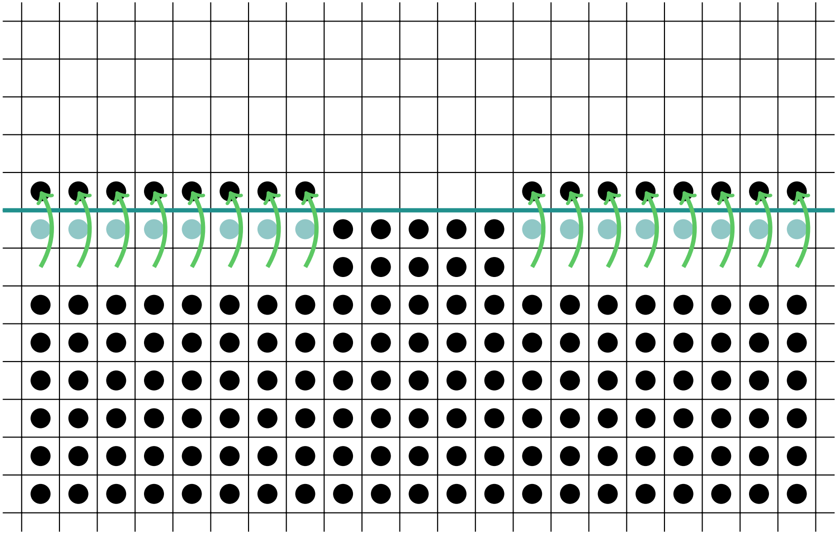

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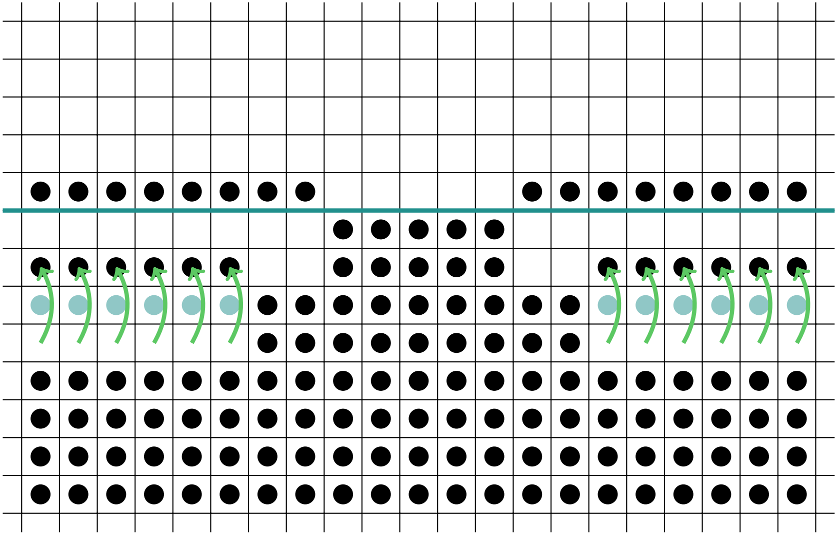

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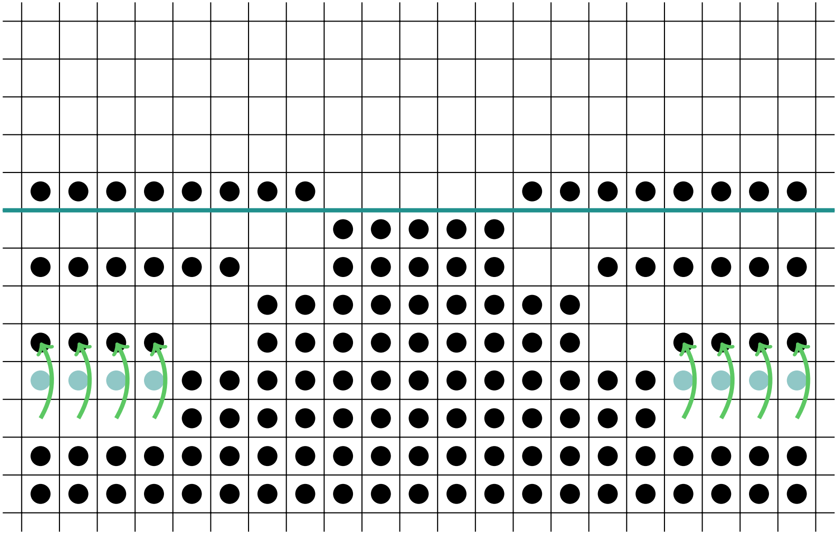

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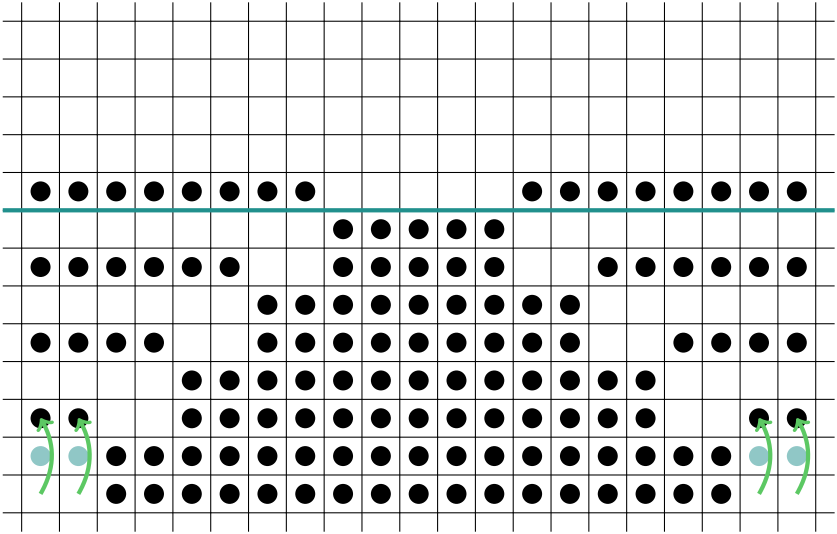

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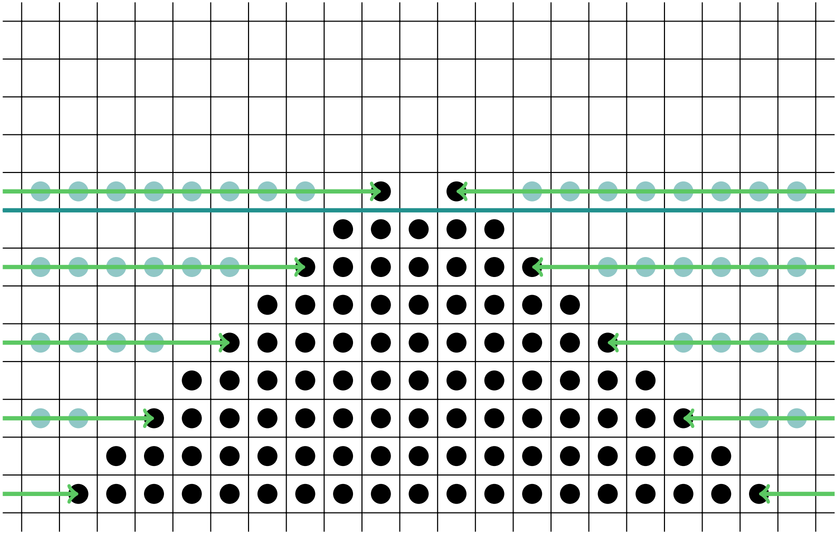
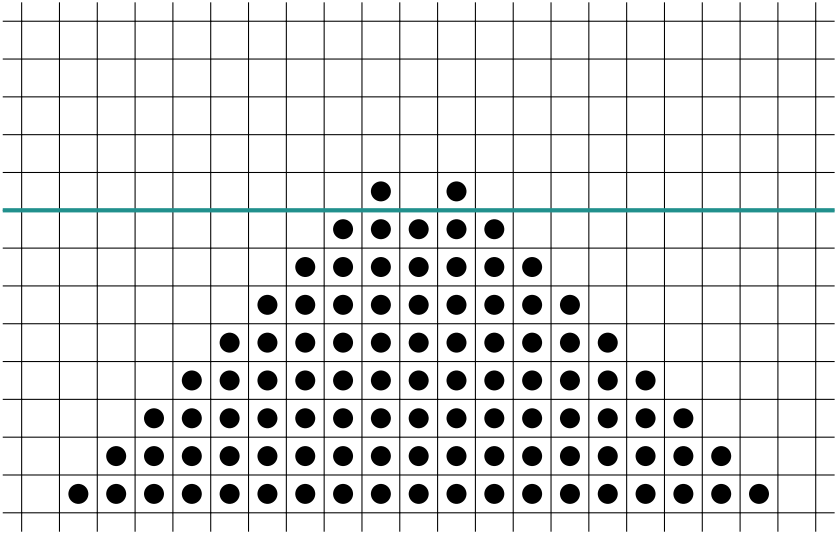
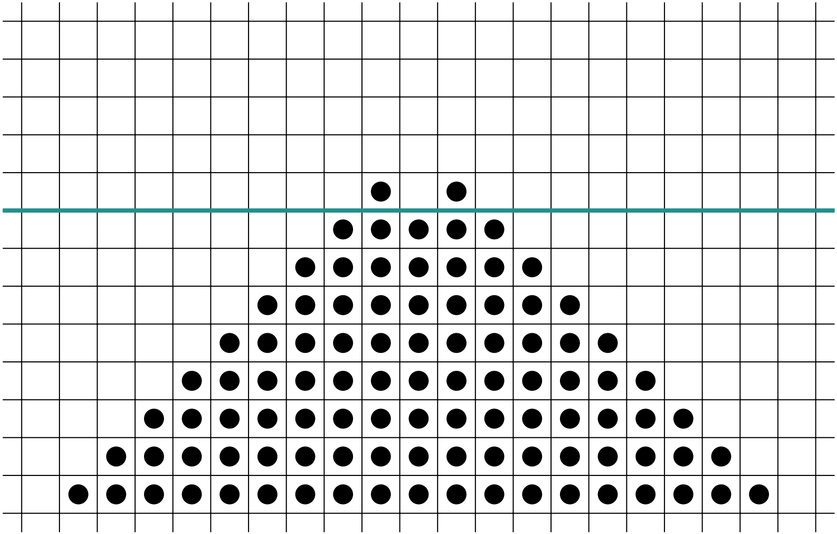


The first half of infinity



Halfway there!

Alan Lee's solution



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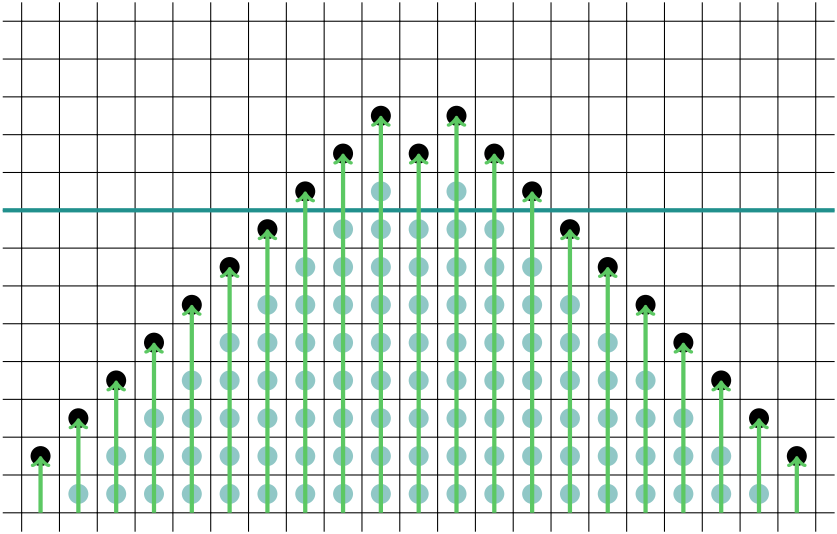

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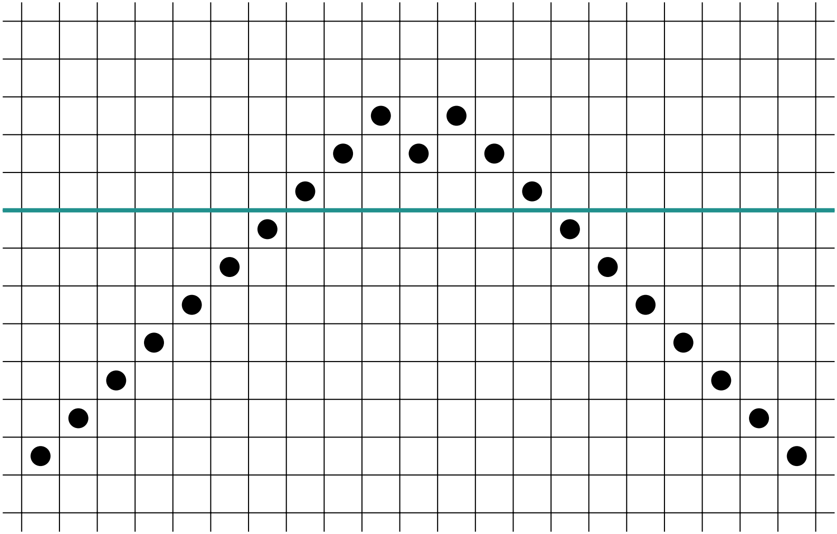

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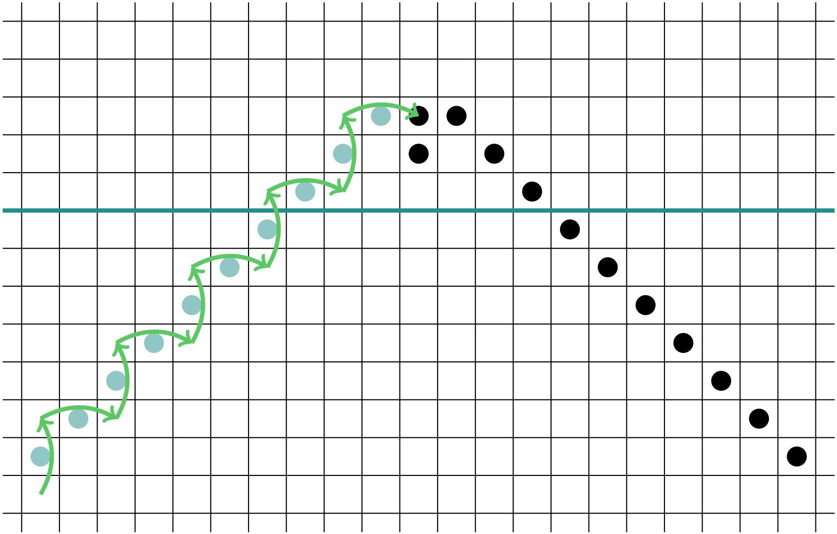

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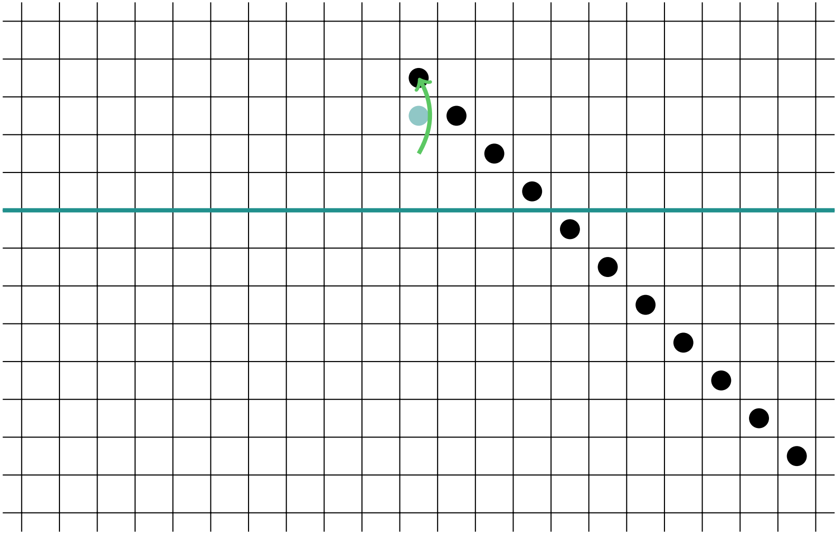

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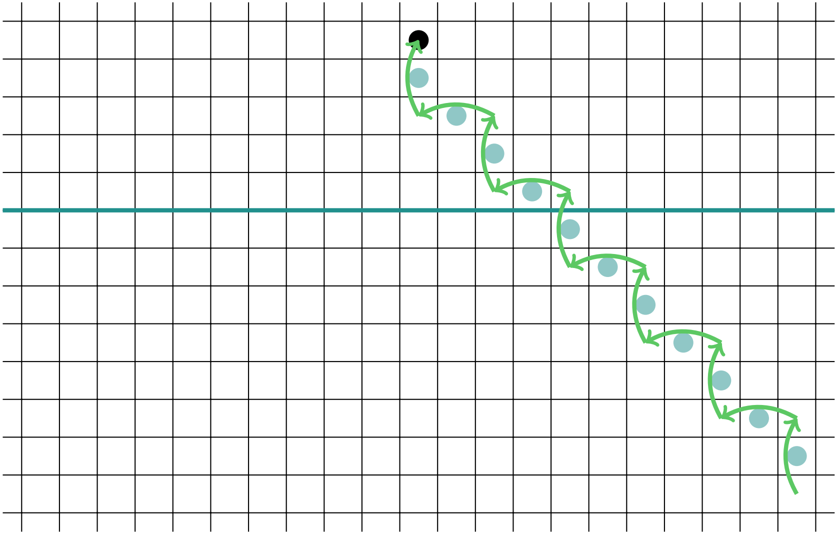

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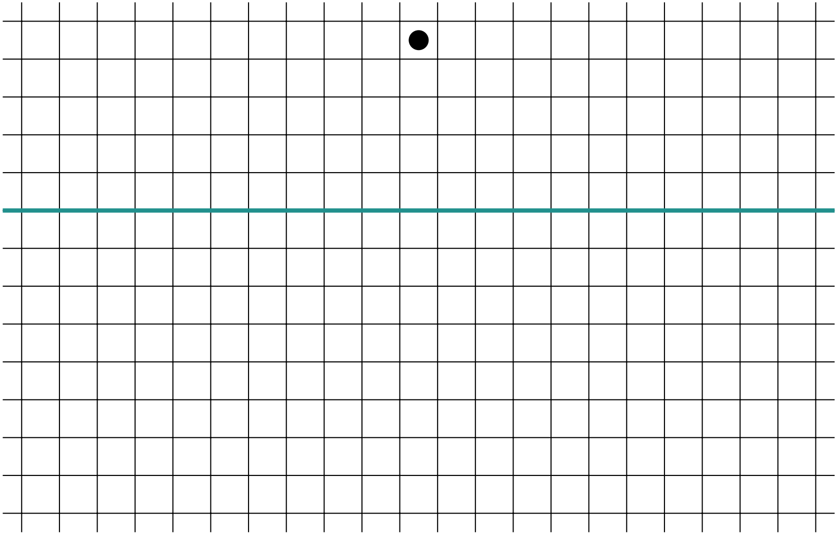

After infinitely many steps, we have reached row 5!

Credits

- Solitaire army was invented by **John Conway** in 1961, who proved that row 5 cannot be reached in finitely many moves.
- **Simon Tatham** first formulated the rules for reaching row 5 in infinitely many moves.
- Simon Tatham and **Gareth Taylor** together found a solution with infinitely many moves.
- **Alan Lee** invented the diagonal whoosh, creating the simplified solution we saw today.

All I did was put together these slides, and make an animation of Alan's solution, which we can go watch at:

`https://misha.fish/solitaire`